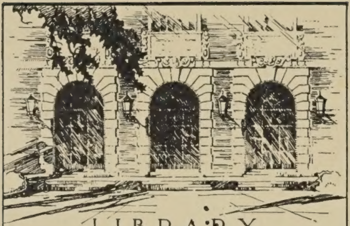

# TABLE OF THE MUTUAL PRODUCTS OF NUMBERS BELOW *ONE HUNDRED.*

##### TABLE OF SQUARES AND CUBES.

| No. | Squar. | Cubes. | No. | Squar. | Cubes. |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 51 | 2601 | 132651 |
| 2 | 4 | 8 | 52 | 2704 | 140608 |
| 3 | 9 | 27 | 53 | 2809 | 148877 |
| 4 | 16 | 64 | 54 | 2916 | 157464 |
| 5 | 25 | 125 | 55 | 3025 | 166375 |
| 6 | 36 | 216 | 56 | 3136 | 175616 |
| 7 | 49 | 343 | 57 | 3249 | 185193 |
| 8 | 64 | 512 | 58 | 3364 | 195112 |
| 9 | 81 | 729 | 59 | 3481 | 205379 |
| 10 | 100 | 1000 | 60 | 3600 | 216000 |
| 11 | 121 | 1331 | 61 | 3721 | 226981 |
| 12 | 144 | 1728 | 62 | 3844 | 238328 |
| 13 | 169 | 2197 | 63 | 3969 | 250047 |
| 14 | 196 | 2744 | 64 | 4096 | 262144 |
| 15 | 225 | 3375 | 65 | 4225 | 274625 |
| 16 | 256 | 4096 | 66 | 4356 | 287496 |
| 17 | 289 | 4913 | 67 | 4489 | 300763 |
| 18 | 324 | 5832 | 68 | 4624 | 314432 |
| 19 | 361 | 6859 | 69 | 4761 | 328509 |
| 20 | 400 | 8000 | 70 | 4900 | 343000 |
| 21 | 441 | 9261 | 71 | 5041 | 357911 |
| 22 | 484 | 10648 | 72 | 5184 | 373248 |
| 23 | 529 | 12167 | 73 | 5329 | 389017 |
| 24 | 576 | 13824 | 74 | 5476 | 405224 |
| 25 | 625 | 15625 | 75 | 5625 | 421875 |
| 26 | 676 | 17576 | 76 | 5776 | 438976 |
| 27 | 729 | 19683 | 77 | 5929 | 456533 |
| 28 | 784 | 21952 | 78 | 6084 | 474552 |
| 29 | 841 | 24389 | 79 | 6241 | 493039 |
| 30 | 900 | 27000 | 80 | 6400 | 512000 |
| 31 | 961 | 29791 | 81 | 6561 | 531441 |
| 32 | 1024 | 32768 | 82 | 6724 | 551368 |
| 33 | 1089 | 35937 | 83 | 6889 | 571787 |
| 34 | 1156 | 39304 | 84 | 7056 | 592704 |
| 35 | 1225 | 42875 | 85 | 7225 | 614125 |
| 36 | 1296 | 46656 | 86 | 7396 | 636056 |
| 37 | 1369 | 50653 | 87 | 7569 | 658503 |
| 38 | 1444 | 54872 | 88 | 7744 | 681472 |
| 39 | 1521 | 59319 | 89 | 7921 | 704969 |
| 40 | 1600 | 64000 | 90 | 8100 | 729000 |
| 41 | 1681 | 68921 | 91 | 8281 | 753571 |
| 42 | 1764 | 74088 | 92 | 8464 | 778688 |
| 43 | 1849 | 79507 | 93 | 8649 | 804357 |
| 44 | 1936 | 85184 | 94 | 8836 | 830584 |
| 45 | 2025 | 91125 | 95 | 9025 | 857375 |
| 46 | 2116 | 97336 | 96 | 9216 | 884736 |
| 47 | 2209 | 103823 | 97 | 9409 | 912673 |
| 48 | 2304 | 110592 | 98 | 9604 | 941192 |
| 49 | 2401 | 117649 | 99 | 9801 | 970299 |
| 50 | 2500 | 125000 | | | |

## EXPLANATION OF THIS ENLARGED MULTIPLICATION TABLE.

The numbers as far as 100, except the first decad, which is omitted as unnecessary, are ranged horizontally in their order from left to right; and their products by like numbers, from 11 to 99 inclusive, which descend along the right hand side, are placed in the corresponding vertical columns. If the numbers to be multiplied be equal, their product, which is in this case a square, will occupy the exterior diagonal; but if those numbers be unequal, the greater is to be sought in the row along the top of the Table, and then glancing down the column to where a line from the other number on the right hand crosses it, the product is found. Thus, the square of 34 is 1156, which stands at the bottom of its column; and to multiply 78 by 62, cast the eye along the column under 78 till opposite to 62 on the side, and the product 4836 occurs.

*Example of Common Multiplication.*

683214
359671
994
2272
4828
1544
3072
6528
490
1120
2380
245732262594

*Example of the Squaring of a Number.*

7912
7912
144
1896
6241
62599744; being the square of the mean diameter of our globe, in English miles.

*Example of Abbreviated Decimal Multiplication.*

62599744
.785398
4836
4602
7566
34
5286
3127
52
6076
59
49165714; which must consequently denote, in English square miles, the fourth part of the surface of the earth.

*Example of Extraction of the Square Root.*

11776237604(343098
6830)211623
30)2040
900
686098)67237604
98)6664
5880
9604

*Example of Extraction of the Cube Root.*

4489
5
13467
6231
961
135294061
31
304957115891(6731
300763
4194115891
403
1612
2914
1891

We may observe, that this table contains 14,052 digits; but had it commenced at zero, it would have contained 16,638. In this aggregate, the several digits recur in very different proportions; of the series 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, the corresponding numbers of recurrences are 2101, 1966, 2388, 1548, 1854, 1577, 1493, 1261, 1534 and 916.

| 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 880 | 891 | 902 | 913 | 924 | 935 | 946 | 957 | 968 | 979 | 990 | 1001 | 1012 | 1023 | 1034 | 1045 | 1056 | 1067 | 1078 | 1089 | 11 |
| 960 | 972 | 984 | 996 | 1008 | 1020 | 1032 | 1044 | 1056 | 1068 | 1080 | 1092 | 1104 | 1116 | 1128 | 1140 | 1152 | 1164 | 1176 | 1188 | 12 |
| 1040 | 1053 | 1066 | 1079 | 1092 | 1105 | 1118 | 1131 | 1144 | 1157 | 1170 | 1183 | 1196 | 1209 | 1222 | 1235 | 1248 | 1261 | 1274 | 1287 | 13 |
| 1120 | 1134 | 1148 | 1162 | 1176 | 1190 | 1204 | 1218 | 1232 | 1246 | 1260 | 1274 | 1288 | 1302 | 1316 | 1330 | 1344 | 1358 | 1372 | 1386 | 14 |
| 1200 | 1215 | 1230 | 1245 | 1260 | 1275 | 1290 | 1305 | 1320 | 1335 | 1350 | 1365 | 1380 | 1395 | 1410 | 1425 | 1440 | 1455 | 1470 | 1485 | 15 |
| 1280 | 1296 | 1312 | 1328 | 1344 | 1360 | 1376 | 1392 | 1408 | 1424 | 1440 | 1456 | 1472 | 1488 | 1504 | 1520 | 1536 | 1552 | 1568 | 1584 | 16 |
| 1360 | 1377 | 1394 | 1411 | 1428 | 1445 | 1462 | 1479 | 1496 | 1513 | 1530 | 1547 | 1564 | 1581 | 1598 | 1615 | 1632 | 1649 | 1666 | 1683 | 17 |
| 1440 | 1458 | 1476 | 1494 | 1512 | 1530 | 1548 | 1566 | 1584 | 1602 | 1620 | 1638 | 1656 | 1674 | 1692 | 1710 | 1728 | 1746 | 1764 | 1782 | 18 |
| 1520 | 1539 | 1558 | 1577 | 1596 | 1615 | 1634 | 1653 | 1672 | 1691 | 1710 | 1729 | 1748 | 1767 | 1786 | 1805 | 1824 | 1843 | 1862 | 1881 | 19 |
| 1600 | 1620 | 1640 | 1660 | 1680 | 1700 | 1720 | 1740 | 1760 | 1780 | 1800 | 1820 | 1840 | 1860 | 1880 | 1900 | 1920 | 1940 | 1960 | 1980 | 20 |
| 1680 | 1701 | 1722 | 1743 | 1764 | 1785 | 1806 | 1827 | 1848 | 1869 | 1890 | 1911 | 1932 | 1953 | 1974 | 1995 | 2016 | 2037 | 2058 | 2079 | 21 |
| 1760 | 1782 | 1804 | 1826 | 1848 | 1870 | 1892 | 1914 | 1936 | 1958 | 1980 | 2002 | 2024 | 2046 | 2068 | 2090 | 2112 | 2134 | 2156 | 2178 | 22 |
| 1840 | 1863 | 1886 | 1909 | 1932 | 1955 | 1978 | 2001 | 2024 | 2047 | 2070 | 2093 | 2116 | 2139 | 2162 | 2185 | 2208 | 2231 | 2254 | 2277 | 23 |
| 1920 | 1944 | 1968 | 1992 | 2016 | 2040 | 2064 | 2088 | 2112 | 2136 | 2160 | 2184 | 2208 | 2232 | 2256 | 2280 | 2304 | 2328 | 2352 | 2376 | 24 |
| 2000 | 2025 | 2050 | 2075 | 2100 | 2125 | 2150 | 2175 | 2200 | 2225 | 2250 | 2275 | 2300 | 2325 | 2350 | 2375 | 2400 | 2425 | 2450 | 2475 | 25 |
| 2080 | 2106 | 2132 | 2158 | 2184 | 2210 | 2236 | 2262 | 2288 | 2314 | 2340 | 2366 | 2392 | 2418 | 2444 | 2470 | 2496 | 2522 | 2548 | 2574 | 26 |
| 2160 | 2187 | 2214 | 2241 | 2268 | 2295 | 2322 | 2349 | 2376 | 2403 | 2430 | 2457 | 2484 | 2511 | 2538 | 2565 | 2592 | 2619 | 2646 | 2673 | 27 |
| 2240 | 2268 | 2296 | 2324 | 2352 | 2380 | 2408 | 2436 | 2464 | 2492 | 2520 | 2548 | 2576 | 2604 | 2632 | 2660 | 2688 | 2716 | 2744 | 2772 | 28 |
| 2320 | 2349 | 2378 | 2407 | 2436 | 2465 | 2494 | 2523 | 2552 | 2581 | 2610 | 2639 | 2668 | 2697 | 2726 | 2755 | 2784 | 2813 | 2842 | 2871 | 29 |
| 2400 | 2430 | 2460 | 2490 | 2520 | 2550 | 2580 | 2610 | 2640 | [illegible] | [illegible] | 2730 | 2760 | 2790 | 2820 | 2850 | 2880 | 2910 | 2940 | 2970 | 30 |

| 99 | |
|---|---|
| 3069 | 31 |
| 3168 | 32 |
| 3267 | 33 |
| 3366 | 34 |
| 3465 | 35 |
| 3564 | 36 |
| 3663 | 37 |
| 3762 | 38 |
| 3861 | 39 |
| 3960 | 40 |
| 4059 | 41 |
| 4158 | 42 |
| 4257 | 43 |
| 4356 | 44 |
| 4455 | 45 |
| 4554 | 46 |
| 4653 | 47 |
| 4752 | 48 |
| 4851 | 49 |
| 4950 | 50 |
| 49 | 51 |
| 48 | 52 |
| 47 | 53 |
| 46 | 54 |
| 45 | 55 |
| 4 | 56 |
| 3 | 57 |
| 2 | 58 |
| 1 | 59 |
| 0 | 60 |
| 9 | 61 |
| 8 | 62 |
| 7 | 63 |
| 6 | 64 |
| 5 | 65 |
| 4 | 66 |
| 3 | 67 |
| 2 | 68 |
| 1 | 69 |
| 0 | 70 |
| [illegible] | 71 |

THE

# PHILOSOPHY

OF

# ARITHMETIC;

EXHIBITING

A PROGRESSIVE VIEW

OF THE

THEORY AND PRACTICE OF CALCULATION,

WITH

AN ENLARGED TABLE OF THE PRODUCTS OF NUMBERS
UNDER *ONE HUNDRED.*

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BY

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EDINBURGH:

*Printed by Abernethy & Walker,*

FOR ARCHIBALD CONSTABLE AND COMPANY, EDINBURGH;
AND LONGMAN, HURST, REES, ORME, AND BROWN
LONDON.

1817.

# PREFACE.

I now discharge my promise, by the publication of a volume, in which Arithmetic is deduced from its principles, and treated as a branch of liberal education. The object proposed was not merely to teach the rules of calculation, but to train the young student to the invaluable habit of close and patient investigation. I have therefore preferred the analytical mode of advancing, and have pursued a route entirely different from that which is followed by the common treatises of arithmetic. In seeking to unfold the natural progress of discovery, I have traced the science of numbers, through the succession of ages, from its early germs, till it acquired the strength and expansion of full maturity. This species of history, combining solid instruction with curious details, cannot fail to engage the attention of inquisitive readers.

If the execution of this little work, which is the result of very considerable research, should at all correspond with the importance of its design, it may supply a capital defect in our systems of public instruction. A great portion of the materials which compose it has already been given to the world, through the wide circulation of the Supplement to the Encyclopædia Britannica; but the distinct and improved form in which it now appears is alone suited for general use. Some persons will perhaps complain that it takes a wider scope than might answer all the common objects of tuition. But there is yet no royal road to knowledge, and whatever is acquired without effort becomes quickly effaced from the memory. Nothing indeed can be more fallacious than to expect any solid or lasting advantage from the substitution of a concise and mechanical procedure. The time required for the study of this treatise could scarcely be more beneficially employed, since it will not only rivet in the mind of the pupil the theory and enlarged practice of calculation, but invigorate, by proper exercise, his reasoning faculties, and consequently prepare him either for entering the labyrinths of business, or engaging in the higher pursuits of science.

College of Edinburgh,
15th October 1817.

# INTRODUCTION.

The idea of number, though not the most easily acquired, remounts to the earliest epochs of society, and must be nearly coëval with the formation of language. The very savage, who draws from the practice of fishing or hunting a precarious support for himself and family, is eager, on his return home, to count over the produce of his toilsome exertions. But the leader of a troop is obliged to carry farther his skill in numeration. He prepares to attack a rival tribe, by marshalling his followers; and, after the bloody conflict is over, he reckons up the slain, and marks his unhappy and devoted captives. If the numbers were small, they could easily be represented by very portable emblems, by round pebbles, by dwarf-shells, by fine nuts, by hard grains, by small beans, or by knots tied on a string. But to express the larger numbers, it became necessary, for the sake of distinctness, to place those little objects or counters in regular rows, which the eye could comprehend at a single glance; as, in the actual telling of money, it would soon have become customary to dispose the rude counters, in two, three, four, or more ranks, according as circumstances might suggest. The attention of the reckoner would then be less distracted, resting chiefly on the number of marks presented by each separate row.

Language insensibly moulds itself to our wants. But it was impossible to furnish a name for each particular number: No invention could supply such a multitude of words as would be required, and no memory could ever retain them. The only practical mode of proceeding, was to have recourse, as on other occasions, to the powers of CLASSIFICATION. By conceiving the individuals of a mass to be distributed into successive ranks and divisions, a few component terms might be made sufficient to express the whole. We may discern around us traces of the progress of numeration, through all its gradations.

The earliest and simplest mode of reckoning was by *pairs*, arising naturally from the circumstance of both hands being employed in it, for the sake of expedition. It is now familiar among sportsmen, who use the names of *brace* and *couple*, words that signify *pairing* or *yoking*.—To count by *threes* was another step, though not practised to an equal extent. It has been preserved, however, by the same class of men, under the term *leash*, meaning the strings by which *three* dogs and no more can be held at once in the hand. —The numbering by *fours*, has had a more extensive application: It was evidently suggested by the custom of taking, in the rapid tale of objects, a pair in each hand. Our fishermen, who generally count in this way, call every *double pair* of herrings, for instance, a *throw* or *cast*; and the term *warp*, which, from its German origin, has exactly the same import, is employed to denote *four*, in various articles of trade.

These simple arrangements would, on their first application, carry the power of reckoning but a very little way. To express larger numbers, it became necessary to renew the process of classification; and the ordinary steps by which language ascends from particular to general objects, might point out the right path of proceeding. A collection of *individuals* forms a *species;* a cluster of *species* makes a *genus;* a bundle of *genera* composes an *order;* a group of *orders* constitutes a *class;* and an aggregation of classes may complete a *kingdom.* Such is the method indispensably required in framing the successive distribution of the almost unbounded subjects of Natural History.

In following out the classification of numbers, it seemed easy and natural, after the first step had been made, to repeat the same procedure. If a heap of pebbles were disposed in certain rows, it would evidently facilitate their enumeration, to break down each of those rows into similar parcels, and thus carry forward the successive subdivision till it stopped. The heap, so analysed by a series of partitions, might then be expressed with a very few low numbers, capable of being distinctly retained. The particular system adopted for this decomposition would soon become clothed in terms borrowed from the vernacular idiom.

Let us endeavour to trace the steps by which a child or a savage, prompted by native curiosity, would proceed in classing, for instance, *twenty-three* similar objects.—

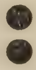
1. He might be conceived to arrange them by successive *pairs.* Selecting *twenty-three* of the smallest shells or grains he could find, he might dispose these in *two* rows, containing each *eleven* counters, and *one* over.

. . . . . . . . . . .
. . . . . . . . . . .   .

Having thus reduced the number to *eleven,* he might sub-divide this again, by representing only one of the rows, with shells *twice* as large as before. He would consequently obtain two rows of *five* each, with an excess of *one*. Instead

of these shells, were he to employ shells of a *double* size, it would be sufficient to denote one of the rows, or to dispose it into *two* rows. These rows con-

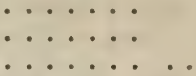
tain each only *two* counters, with *one* remaining. Again, by adopting counters of a double size, the last row might be represented by one pair, each contain-



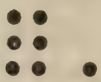
ing only *two* marks exactly. These again could be denoted by a *single* counter, of *twice* the former dimensions.—Hence the number *twenty-three*, as

decomposed by repeated *pairing*, would be denoted by *one* counter of the fifth order, *one* of the third, *one* of the second, and another of the first.

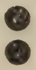
2. Again, suppose a person should attempt to represent the same number, by *triple* rows of shells or counters. He would first have *seven* of the smallest shells in each row, with



an excess of *two*. Then, expressing one of the rows by shells of *three* times the size or value, it would be again resolved into *three*

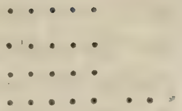


rows, each containing *two* counters, with an excess of *one*. But these rows might be represented by *two* counters of *triple size*;



and here the decomposition stops. The number *twenty-three* is thus expressed, on the system of *triplication*, by *two counters* of the third order, *one* of the second, and *two* of the first.

3. Lastly, conceive the number *twenty-three* to be reckoned by *double pairs* or *quadruple* rows. Each row would now contain only *five* counters, with an excess of *three*.



But a *single row* would express the same as all these *four*, if each counter in it were changed into another of quadruple size or value; and, consequently, this row might be again distributed into *four* ranks, each consisting of *a single* mark, with *one* left.

Retain one of these ranks, and substitute a counter of *quadruple effect*, which will express the whole amount.

Hence *twenty-three*, analysed by the system of *double pairs* or *warps*, would be represented by *one* counter of the third order, *one* of the second, and *three* of the first.—

It is easy to conceive how, in other cases, the process of decomposition could be carried forward.

In the ruder periods of society, a gradation of counters, accommodated to such a process of numerical analysis, was supplied by grains, beans, pebbles, or shells of different sizes. The series of such natural emblems, however, is very limited, and would soon confine the range of decomposition. To obtain a greater extent, it was necessary to proceed by a swifter analysis; to distribute the counters, for instance, successively into ten or twenty rows, and to make pebbles, shells, or other marks, having their size only doubled perhaps or tripled, to represent values increased ten or twenty fold. Beyond this stage in the progress of numeration, none of the various tribes dispersed over the vast American Continent seem ever to have passed. In the Old World, it is probable that a long pause of improvement had ensued among the nations which were advanced to the same point in the arts of life. But the necessity, in such arithmetical notation, of employing the natural objects to signify a great deal more than their relative size imports, would lead at last to a most important step in the ascent. Instead of distinguishing the different orders of counters by their *magnitude*, they might be made to derive an *artificial value* from their *rank* alone. It would be sufficient, for that purpose, to employ marks all of the same kind, only disposed on a graduating series of vertical bars or columns. The augmented value which these marks acquire in rising through the successive bars, would evidently be quite arbitrary, depending, in every case, on a key to be fixed by convention. This point in the chain of discovery was attained by the Greeks at a very early period, and communicated to the Romans, who continued, during their whole career of empire, to practise a sort of tangible arithmetic, which they transmitted to their successors in modern Europe.

Such humble expedients might suffice for all the computations required in the ruder periods of society. But men were insensibly led to frame permanent symbols to denote numbers by that feeling, which unceasingly prompts them to seek the approbation and applause of their fellows. The aspiring leader of a successful band, or the petty legislator of a rising community, is anxious to preserve the memory of the exploits he performed, or the benefits he conferred. He is not content with obtaining the applause of his contemporaries; this fleeting existence is insufficient to fill his imagination; he looks anxiously beyond the grave, and sighs for the admiration of generations yet unborn. Hence the anxiety among all people to erect monuments of high achievements or illustrious characters. In the early periods of society, a vast mound of earth, or a huge block of stone, was the only memorial of any great event. But, after the simpler arts came to be known, efforts were made to transmit to posterity the representations of the objects themselves. Sculptures of the humblest kind occur on monumental stones in all parts of the world, sufficient to convey tolerably distinct images of the usual occupation and employments of the personages so commemorated.

The next step in the progress of society was to reduce and abridge those rude sculptures, and thence form combinations of figures approaching to the hieroglyphical characters. At this epoch of improvement, the first attempts to represent numerals would be made. Instead of repeating the same objects, it was an obvious contrivance to annex to the mere individual the simpler marks of such repetition. Those marks would of necessity be suited to the nature of the materials on which they were inscribed, and the quality of the instruments employed to trace them. In the historical representations, for instance, which the Mexicans and certain Tartar hordes painted on skins, a small coloured circle, as exhibiting the original counter, shell, or pebble, was repeated to denote numbers. But, on the Egyptian Obelisks, the lower numerals at least, are expressed by combined strokes. None but straight lines, indeed, are fitted for being carved on pillars of stone, or cut on the face of wooden posts. Even after the use of Hieroglyphics had been laid aside, and the artificial system of alphabetic characters adopted, the rectilineal forms were still preferred; as evidently appears in the Greek and Roman capitals, which, being originally of the lapidary sort, are much older than the small or current letters. The Runic letters, in which the Northern Languages of Europe were at first engraved, consist almost entirely of simple strokes inserted at different angles.

The primary numeral traces may, therefore, be regarded as the commencement of a philosophical and universal character, drawn from nature itself, and alike intelligible to all ages and nations. They are still preserved, with very little change, in the Roman notation; consisting of simple strokes variously combined, which were imported perhaps before the adoption of the alphabet itself by the Grecian colonies that settled in Italy, and gave rise to the Latin name and commonwealth. The lower classifications of numbers had gradually fallen into disuse, and given place to the more expeditious and convenient system of advancing by successive tens, which arose very naturally indeed from the practice common among rude people, of counting by their fingers on both hands. Assuming, therefore, a perpendicular line | to signify *one*, another such || would express *two*, the junction of a third ||| *three*, and so repeatedly till the reckoner had reached *ten*. The first class was now completed, and to intimate this, he threw a dash across the stroke or common unit; that is, he employed two decussating strokes X to denote *ten*. He next repeated this mark, to express *twenty*, *thirty*, and so forth, till he had finished the second class of numbers. Arrived at *an hundred*, he would signify it, by joining another dash to the mark for ten, or by merely connecting three strokes thus ⊏. Again, the same spirit of invention might lead him to repeat this character, in denoting *two hundred*, *three hundred*, and so forth, till the *third* class was completed. *A thousand*, which begins the fourth class on the ordinary mode of numeration, was therefore expressed by four combined strokes M; and this was the utmost length to which the Romans first proceeded by direct notation.

But the division of these marks afterwards furnished characters for the intermediate numbers, and thence greatly shortened the repetition of the lower ones. Thus, having parted in the middle the two decussating strokes X denoting *ten*, either the under half Λ, or the upper half V, was employed to signify *five*. Next, the mark ⊏ for an hundred, consisting of a triple stroke, was largely divided into ┌ and └, either of which represented *fifty*. Again, the four combined strokes M, which originally formed the character for *a thousand*, came afterwards, in the progress of the arts, to assume a round shape ↀ, frequently expressed thus CIↃ, by two disparted semicircles divided by a diameter: This last form, by abbreviation on either side, gave two portions CI and IↃ to represent *five hundred*.

The process appears easy, therefore, to devise an universal character for expressing numbers. But it was a very different task, to reduce the exhibition of language in general to such concise philosophical principles. This attempt seems accordingly to have been early abandoned by all nations, except the Chinese. The inestimable advantage of uniting together the whole human race, in spite of the diversity of tongues, by the same permanent system of communication, was sacrificed for the simpler attainment, of representing, by artificial signs, those elementary and fugitive sounds, into which the words of each particular dialect could be resolved. Hence the ALPHABET was invented, which, notwithstanding its obvious defects, must ever be regarded as the finest and happiest effort of genius.

More letters were afterwards added in succession, as the analysis of the primary sounds became more complete; but the alphabet had very nearly attained its present form, at the period when the Roman commonwealth was extending its usurpation over Italy. About that epoch, a sort of re-action seems to have arisen between the artificial and the natural systems, and the numeral strokes were finally dis-placed by such alphabetic characters as most resembled them. The ancient Romans employed the letter I, to re-present the single stroke or mark for *one;* they selected the letter V, since it resembles the upper half of the two de-cussating strokes, or symbol for *five;* the letter X exactly depicted the doubled mark for *ten;* again, the letter L was adopted, as resembling the divided symbol for *fifty;* while the entire symbol, or the tripled stroke, denoting an *hun-dred,* was exhibited by the hollow square ⊏, the origi-nal form of the letter C before it became rounded over. The quadrupled stroke for a *thousand* was distinctly repre-sented by the letter M, and its variety by the compound character cIↄ, consisting of the letter I inclosed on both sides by C and by the same letter reversed; the latter portion of this again, or Iↄ, being condensed into the let-ter D, expressed *five hundred.*

The Greeks, after having communicated to the founders of Rome the elements of the numeral characters, which are still preserved, again exercised their inventive genius in framing new systems of notation. Discarding the sim-ple original strokes, they sought to draw materials of con-struction from their extended alphabet. They had no fewer than three different modes of proceeding. 1. The letters of the alphabet, in their natural succession, were employed by them to signify the smaller ordinal numbers. In this way, for instance, the books of Homer's *Iliad* and *Odyssey* are usually marked. 2. The first letters of the words for numerals were adopted as abbreviated symbols. A simple and ingenious device was used for augmenting the powers of those symbols: any letter inclosed by a line on each side, and another drawn over the top thus |¯|, being made to signify *five thousand* times more. 3. But a mighty stride was afterwards made in numerical notation by the Greeks, when they distributed the twenty-four letters of their alphabet into three classes, corresponding to *units*, *tens*, and *hundreds*. To complete the symbols for all the *nine* digits, an additional appropriate character was introduced in each class.

This beautiful system was vastly superior in clearness and simplicity to the combinations of strokes, retained by the Romans, and transmitted by them to the nations of modern Europe. It was even tolerably fitted as an instrument of calculation, to which the Roman numerals were totally inapplicable.

The Greek notation proceeded directly as far as *nine hundred and ninety-nine*; but, by subscribing an *iôta* or short dash under any character, its value was augmented a *thousand* fold; or by writing the initial letter of *myriad*, the effect was increased still *ten times more*. With the help of punctuated letters, therefore, it reached to *ten thousand*, comprising four terms of our ordinary scale; and by means of the subscribed M, it was carried over another similar period, or fitted to express a *hundred million*. But the penetrating genius of Archimedes quickly discerned the powers, and unfolded the properties, of such progressions. He took the square of the limit of the common numeral system, or an *hundred millions*, being *ten-*

*thousand* times *ten-thousand*, for the index of a new scale of arrangement, which therefore advanced by strides *eight* times faster than the simple denary notation. This comprehensive series he proposed to carry as far as eight periods, which would hence correspond to a number expressed in our mode by *sixty-four* digits.

Apollonius, who, next to the Sicilian philosopher, was the most ingenious and inventive of all the ancient mathematicians, resumed that scheme of numeration which had been suffered to lie neglected, simplified the construction of its scale, and reduced it to a commodious practice. For greater convenience, he preferred the *simple myriad* as the root of the system, which therefore proceeded by successive periods, corresponding to *four* of our digits. These periods were distinguished by breaks or blanks.

The learned and profound astronomer Ptolemy modified this system in its descending range, by applying it to the sexagesimal subdivisions of the lines inscribed in a circle. He likewise advanced a most important step, by employing a small or accentuated *o*, to supply the place of any number wanting in the order of progression.

The Arithmetic of the Greeks, thus successively moulded by the ingenuity of their great Geometers, had attained a singular degree of perfection, and was capable, notwithstanding its cumbrous structure, of performing operations of very considerable difficulty and magnitude. But those masters of science, rich in their mental resources, overlooked the advantages resulting from a simpler mode of arrangement. They had only to ascend more slowly, and proceed by *tens* instead of periods of *myriads*; that is, to retain as numerals no more than the first set of their alphabetic characters, which were already employed with a point or a short dash subscribed to denote *thousands*. This might seem an easy step in the progress of invention, but the current of ideas had already flowed beyond it. Nor during the ebbing tide that preceded the fatal extinction of science among the Greeks, was any farther simplification effected, which would have shed a pale ray over the evening of that Philosophy which was again destined to emerge from the thickest darkness, and relume the world. For the knowledge of our system of elementary numerals, which may be justly styled the Alphabet of Arithmetic, we are indebted to a people extremely inferior to those instructors of mankind, in genius, acuteness, and general energy of character. Whether the Hindūs lighted on that happy contrivance themselves, or derived it from their communication with the natives of Upper Asia, there is yet no sufficient evidence to decide. They seem, however, to have become acquainted with it nearly two thousand years ago, and to have thenceforth commonly employed that mode of notation. From the Hindūs, again, their Arabian conquerors appear, about the ninth century of our æra, to have received an improvement at once so simple and important. These industrious cultivators of science afterwards imparted the valuable present to their countrymen the Moors, who still occupied the finest portion of Spain. From this centre, it was gradually communicated over Europe. The earliest traces of the numeral characters among the Christians may be referred to the end of the thirteenth century. They were at first introduced only into almanacs and astronomical tables; but their great convenience soon brought them into more general use. Originally those characters were somewhat differently shaped from the present; they had attained, however, their actual forms not long after the invention of the noble art of printing, or about the commencement of the sixteenth century. The digits were now adopted almost universally throughout Europe into the practice of Arithmetic. Yet vestiges still remained of the performing of numerical operations by means of counters and other palpable emblems. Nor is this rude mode of computation quite so contemptible as would at first appear, since the Chinese continue to employ it with success in all their mercantile transactions. At any rate, it deserves attention, both for its connection with the history of science, and its importance in illustrating the properties of numbers, and explaining the grounds of calculation.

We shall therefore view ARITHMETIC under two very distinct forms, that would require separate appellations. 1. *Palpable Arithmetic*, in which the numbers are exhibited by counters, or abbreviated representatives of the objects themselves; and, 2. *Figurate Arithmetic*, in which the numbers are denoted by help of certain symbols, or artificial characters, disposed after a particular order. The progress of Arithmetic is analogous to that of writing, but it has followed the advances and transitions of this sublime art at a great distance. The numeration by counters, balls or strokes, evidently resembles hieroglyphics or picture writing; while the invention of the alphabet, so happily contrived for the rapid transmission of thought, probably led the way to the subsequent discovery of the science of *Figurate Arithmetic*, founded nearly on similar principles. These capital divisions of Arithmetic we shall consider in succession, pointing out the application of each mode to *Numeration*, and to the several depending operations of *Addition*, *Subtraction*, *Multiplication*, and *Division*.

# PALPABLE ARITHMETIC.

## NUMERATION.

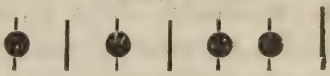
Suppose the objects to be reckoned were so numerous, that *eighty-six* counters might be required to represent them. Placed in a single row, these counters would only give the very confused idea of multitude. But, if counted by *pairs*, or divided into two rows of *forty-three* each, they would become a little more distinct. Were every counter now, in each row, to denote a *pair*, a single row of them would have the power of both. Let this row be reckoned again by pairs, and it will change into two higher rows, each consisting of *twenty-one* counters, with an excess of *one*. But one of these *third* rows would be sufficient alone, if each counter in it were esteemed equal to a *pair* in the *second* rows, or equal to a *duplicate pair*, or *four* in the *first* row. Again, tell one of the *third* rows over by pairs, and the *twenty-one* counters will be converted into two *fourth* rows, containing *ten* each, and *one* over. In like manner, each counter of a *fourth* row, being conceived equal to a pair in the *third* row, or a *triplicate pair*, that is, *eight* on the *first row*; a single row, including *ten* of those higher counters, would have the same effect. Now these *ten* would be reduced to a pair of rows of *five* counters each. Let each counter in the *fifth* row have the power of two in the *fourth* row, or of a *quadruplicate pair*, or *sixteen* in the *first row*; and *five* such counters would be sufficient. But *five* would give *two* pairs of *sixth* rows, one of which might denote the whole, if each counter in it were held equal to *two* in the preceding row, or to a *quintuplicate pair*, or *thirty-two* on the *first row*. Again, the last *two* counters would be divided into a *single pair* of a higher order.

This analysis appears tedious when so detailed, but it would proceed with great ease and rapidity in practice. The number *eighty-six* would, therefore, on the system of successive *pairing*, be expressed by *one sextuplicate pair*, *one quadruplicate pair*, *one duplicate pair*, and *one single pair*. The language appears very uncouth, merely from its novelty and inaptness to our idiom; but its elements are extremely clear and simple. If a few cognate words had been devised to express the several combinations of pairs, or the ascending scale of the powers of two, it would have removed every objection.

This arrangement, whereby a number is analysed into certain elements by the operation of distributing it and its sections into successive *pairs* or *duads*, may be called the *Binary Scale*, of which *two* is the *root* or *index*. This scale, resting on so narrow a basis, expands slowly, and is therefore not very fit for expressing large numbers by words. But it is well adapted for the simplicity of emblematic exhibition. Suppose marks or counters were placed in perpendicular rows or parallel bars, proceeding from the right hand to the left, such that a counter on any bar should be equivalent to *two* laid on the bar immediately below it. Instead of putting the *eighty-six* counters on the first bar, it would be the same thing, to place *forty-*

*three* on the second bar; the effect here again would be the same as to leave *one*, and drop *twenty-one* on the third bar. Counting these *twenty-one* also by pairs, we should place *one* on the third bar, and carry *ten* to the next. But *ten*, divided into pairs, would leave the fourth bar vacant, and throw *five* to the next. The decomposition thus effected would appear as below; where only *four* counters and *three* blanks are sufficient to exhibit the number *eighty-six*. By this elementary arrangement a very distinct idea is conveyed: The eye can easily catch the picture, and the memory preserve it.

BINARY SCALE.



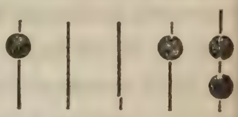
A similar effect would be produced, though much less clearly, by the combination of strokes, like the Runic sculpture, dots being employed to indicate the blanks or vacant terms. — . — . — — .

It is obvious, that this very simple scale would require only one set of marks. These on the ascending bars signify *two*, *four*, *eight*, *sixteen*, &c.; and the number now expressed is divided into *sixty-four*, *sixteen*, *four* and *two*.

Some feeble traces of the *Binary Notation* are found in the early monuments of China. Fouhi, the first Emperor and the founder of that vast monarchy, is venerated in the East as a promoter of Geometry, and the inventor of a science, of which the knowledge has been since lost. The emblem of this occult science appears to consist of eight separate clusters of three parallel lines or *trigrams*, drawn one above another after the Chinese manner of writing, and represented either entire or broken in the middle. Those varied trigrams were called *Koua* or *suspended symbols*, from the circumstance of their being exposed in the public places. In the composition of such clusters we may perceive the application of the *Binary Scale*, carried only to three ranks, or as far as the number *eight*. The entire lines signify *one*, *two* or *four*, according to their order, while the broken lines are void, and serve merely to indicate the rank of the others.

The *Ternary Scale* of numeration, which reckons by successive *threes* or *triads*, advances with more speed. Thus, suppose, as before, that *eighty-six* were to be exhibited on it. Counted by *threes*, or, in the sportsman's phrase, by *leashes*, *two* counters would be left on the first bar, and *twenty-eight* thrown to the next bar, or that of the *simple triads*. These *twenty-eight* counters, being again told by *threes*, would leave *one* on the second bar, and carry *nine* to the third bar, or that of *duplicate triads*. The *nine* reckoned by *three* in succession, would now pass over the third and fourth bars, and throw *one* mark to the *sixth bar*, or that of *quintuple triads*. The original number so decomposed, might therefore be denominated *one quintuple triad*, *one single triad*, and *two*. It is denoted by four counters, as in the mode here annexed. The number might likewise be readily expressed, though less perfectly, by combined strokes and points.

TERNARY SCALE.

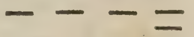


It is apparent that the *Ternary* Scale, though more powerful than the *Binary*, requires two sets of marks or counters. In the example now taken, each counter on the ascending bars represents *three*, *nine*, *twenty-seven*, or *eighty-one*; and the number itself is consequently divided into one *eighty-one*, one *three* and two units.

Let still the same number be arranged on the *Quaternary Scale*, which proceeds by *Fours* or *Tetrads*. *Eighty-six*, told over by *double pairs* or *warps*, would leave *two* counters on the first bar, and carry *twenty-one* to the next. This *twenty-one* again, reckoned by *warps* or *throws*, would drop *one* counter on the second bar, and transfer *five* counters to the third bar. The *five* being now counted, would leave *one* counter on this bar, and carry *one* to the next. The original number would therefore be described as containing *one triplicate tetrads*, *one duplicate tetrads*, *one tetrad and two*; and it would be designated in this manner:

QUATERNARY SCALE.

Or, if the less satisfactory mode of strokes were employed, *one hundred and sixty-five* would be thus exhibited:

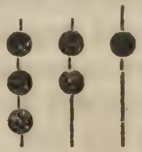


The original number is thus analysed into *once sixty-four*, *once sixteen*, *four* and *two*.—*Three* sets of counters would evidently be required to fit this scale for its application.

The *Quaternary Scale* may be considered as a duplication of the *Binary*, each bar of the former comprising two bars of the latter. It is alleged that the Guaranis and Lulos, two of the very lowest races of savages which inhabit the boundless forests of South America, count only by fours; at least that they express the number *five* by *four* and *one*, *six* by *four* and *two*, and so forth. We may likewise infer from a passage in Aristotle, that a certain tribe of Thracians were accustomed to use the quaternary scale of numeration. If such was the historical fact, that simple race must have not advanced beyond the early practice of reckoning successively by *casts* or *warps*.

To rise a step farther, let the same number be represented on the *Quinary Scale*, which reckons by the series of *fives* or *pentads*. Classed in this way, it would leave *one* on the first bar, and throw *seventeen* counters to the second. Told over again, it would leave *two* counters on the second bar, and carry *three* to the third. *Eighty-six* would therefore be denominated *three duplicate pentads*, *two single pentads*, and *one*. It would thus be denoted:

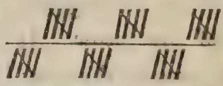
QUINARY SCALE.



This number might also be exhibited on the same scale by a combination of strokes.

By this classification, the number *eighty-six* is divided into *three twenty-fives*, *two fives*, and *one*.—The *root* or *index* of the scale being five, it would require *four* sets of counters to adapt it for practice.

The first bar of the *Quinary Scale* is actually used in this country among wholesale traders. In reckoning articles delivered at the warehouse, the person who takes charge of the *tale*, having traced a long horizontal line, continues to draw, alternately above and below it, a *warp* or *four* vertical strokes, each set of which he crosses by an oblique score, and calls out *tally!* as often as the number *five* is completed.



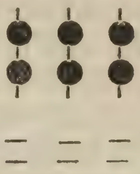
&c.

The *Quinary* system has its foundation in nature, being evidently derived from the practice of counting over the fingers of one hand. It appears accordingly, at a certain stage of society, to have been adopted among different nations. Thus, the Omaguas and the Zamucas of South America reckon generally by *fives*, which they call *hands*.

The Toupinambos, a most ferocious and warlike race that inhabit the wilds of Brazil, would seem, according to the relation of Lery, to use the same kind of numeration. To denominate *six*, *seven* and *eight*, those tribes only add to the word *hand*, the names for *one*, *two*, *three*, &c. The same mode, as we learn from Mungo Park, is practised by some African nations, particularly the Yolofs and Foulahs, who designate *ten* by *two hands*, *fifteen* by *three hands*, and so progressively. The *Quinary Numeration* seems likewise, at a former period, to have obtained in Persia; the word *pentcha*, which denotes *five*, being obviously derived from the radical term *pendj*, which signifies a *hand*.

Suppose *eighty-six* to be now disposed in the *Senary Scale*, which proceeds by successive *sixes* or *hextads*. Parted into *six* rows, that number would leave *two* counters on the first bar, and cast *fourteen* to the next; this *fourteen*, being reckoned by *sixes*, would drop *two* counters on the second bar, and transfer *two* to the third bar. The original number would hence be described as *two duplicate hextads*, *two single hextads* and *two*. It is likewise represented thus by strokes.

SENARY SCALE.

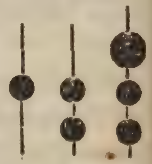


The *Senary* arrangement has few advantages to recommend it; yet it seems at one period to have been adopted in China, at the mandate of a capricious tyrant, who, having conceived an astrological fancy for the number SIX, commanded its several combinations to be used in all concerns of business or learning throughout his vast Empire.

When the index of the progressive scale is larger, it often becomes inconvenient to place so many counters as are wanted on the same bar. But this notation may be abridged in the case where the index is an even number, by adopting a counter of greater dimensions to signify the half of it.

Thus, in the *Octary Scale*, which proceeds by successive *eights* or *octads*, the original number would leave *six* counters on the first bar, and throw *ten* to the next; and this *ten* being told over again by *eights*, would leave *two* counters on the second bar, and carry *two* to the third. If, therefore, the large counter signify half the index, or four, *eighty-six* would be thus denoted.

OCTARY SCALE.



The number is denominated *one duplicate octad*, *two single octads*, and *six*; and it has been decomposed into *one sixty-four*, *two eights*, and *six*.

Suppose now that the *Denary Scale* of Notation were employed. The same number, reckoned by *tens* or *decads*, would leave *six* counters on the first bar, and cast *eight* to the next bar. This arrangement furnishes the denomination of *eight decads*, and *six*; which is simpler than any of the former appellations, and yet it sounds uncouth, owing merely to our want of familiarity with the terms. It would be marked in this way:

DENARY SCALE.

We are thus conducted by successive advances to that system of numeration which has prevailed among all civilized nations, and become incorporated with the very structure of language. This almost universal consent clearly bespeaks the influence of some common principle. Nor is it hard to perceive, that the arrangement of numbers by

TENS would naturally flow from the practice so familiar in the earlier periods of society,—that of counting by the fingers on both hands. The composition of the terms employed in the more polished tongues of antiquity, however, is not easily or clearly traced. But the origin of the names imposed on the radical numbers, appears conspicuously displayed in the nakedness of the savage dialects. The Muysca Indians, who formerly occupied the high plain of Bogota in the province of Grenada, were accustomed to reckon first as far as *ten*, which they called *quihicha* or a *foot*, meaning no doubt the number of toes on both feet, with which they commonly went bare and exposed; and, beyond this number, they used terms equivalent to *foot one*, *foot two*, &c., corresponding to *twelve*, *thirteen*, &c. Another tribe, likewise inhabitants of South America, the Sabiconos, call *ten*, or the root of the scale, *tunca*, and merely repeat the same word to signify an *hundred* and a *thousand*, the former being termed *tunca-tunca*, and the latter *tunca-tunca-tunca*.

Etymology, guided by the spirit of philosophy, furnishes a sure instrument for disclosing the monuments of early conception, preserved, though disguised, in the structure of language. Our own dialect, as immediately derived from its Gothic stem, betrays a composition not less rude or expressive than the simple articulation of the Sabiconos. The words *eleven*, *twelve*, &c. anciently signified merely *one*, *two*, &c. *leave*; intimating no doubt that *one* and *two* are to be *kept*, while *ten*, the root of the scale, is set aside. *Twenty*, *thirty*, &c. meant simply *two*, *three*, &c. *drawings*; importing that so many *tens* are to be taken from the heap. *An hundred*, in the Gothic and Saxon, intimated that *ten* was to be *ten times drawn or told*.

It is remarkable that the Peruvian language was actually richer in the names for numerals than the polished dialects of ancient Rome or Greece. The Romans, we have seen, went not farther than *mille*, a thousand; and the Greeks made no distinctive word beyond *μυρια*, or ten thousand. But the inhabitants of Peru, under the Incas, following the *Denary System*, had the term *huc*, to denote ONE; *chunca*, TEN; *pachac*, ONE HUNDRED; *huaranca*, A THOUSAND; and *hunu*, A MILLION. These words are either original, or have been formed, like our numerical terms, by the abbreviation of certain compound expressions.

It appears, from an early document, that the Indian tribes who surrounded the infant colony of New England, used the *Denary Scale* of arrangement, and had a set of distinct words to express the numbers as far as a *thousand*, and could even advance as high as *one hundred thousand* by help of combined terms. Thus, ONE they named *nquit*; TEN, *piùck*; AN HUNDRED, *pâwsuck*; and A THOUSAND, *mittânug*. But these words are apparently compound, and would doubtless be found to throw much light on the subject of numeration, if we had any means of analysing them. It is likewise asserted, that those savages employed grains of corn for numerical symbols, and were very expert in computation.

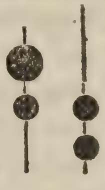
The Laplanders, in their system of numeration, join very significantly the cardinal to the ordinal numbers. Thus, to express ELEVEN and TWELVE, they say *auft nubbe lokkai*, and *gouft nubbe lokkai*; that is, *one to the second* TEN, and *two to the second* TEN. In like manner, to signify TWENTY-ONE and TWENTY-TWO, those rude people use the expressions *auft gooalmad lokkai*, and *gouft gooal-*

*mad lokkai,* meaning *one added to the third* TEN, and *two added to the third* TEN. This procedure affords a curious and very happy illustration of the principle of numeral arrangement.

The *Duodenary System* of arrangement was introduced at a more advanced stage of society. It plainly drew its origin from the observation of the celestial phenomena, there being twelve months or lunations commonly reckoned in a solar year. The Romans likewise adopted that index, to mark their subdivision of the unit of measure or of weight.

The mode of reckoning by *twelves* or *dozens* has been very generally employed in wholesale business. Nor is its application there confined to the first term of the progression, but extends to the second or even the third term. Twelve dozen, or *an hundred and forty-four*, makes the long or *great hundred* of the Northern Nations, or the *Gross* of traders. Twelve times this again, or *seventeen hundred and twenty-eight*, forms the *Double Gross*.

DUODENARY SCALE.



Let the same number, *eighty-six*, as before, be reckoned on the *Duodenary Scale.* It contains *seven dozen* and *two*, and consequently will be represented as here annexed.

Next to the Denary Scale itself, the system of counting by progressive *scores* or *twenties*, derived from the same source, appears to have been the most prevalent. The savage who had reckoned the fingers on both hands, and then the toes on both feet, amounting to *twenty* in all, might seem to have reached the utmost limit of natural calculation. The Guaranis, a very simple and inoffensive tribe, who live on the shores of the Marãnon, are accordingly said to proceed no farther in their direct numeration. When these people want to signify *an hundred*, they only place in a row *five* heaps of maize, each consisting of *twenty* grains. The Mexicans, however, being more advanced in society, were accustomed to employ the higher terms of the same progression, thus combining the *Denary* with the *Binary* scales. In the ancient hieroglyphic paintings of that unfortunate race, *units*, as far as a *score*, are exhibited by *small balls*; and *twenty* is denoted by a figure, which some authors, and particularly Clavigero, have mistaken for a *club*, but which was really a *small standard* or *flag*. In the same curious monuments, *twenty scores*, or *four hundred*, is signified by *a spreading open feather*; probably, because the grains of gold, lodged in the hollow of a quill, represented, in some places, money or the medium of exchange. This symbol has, from the rudeness of the drawing, been taken at times for a pine-apple, an ear of maize, or even the head of a spear; but its application to intimate the *duplicate scores* is certain and invariable. A sack or bag was also painted by those ingenious people, to represent *twenty times twenty scores*, or *eight thousand*: It was of the same form as a purse called *xiquipilli*, and supposed to hold *eight thousand* grains of cacao. The annexed figures, copied from Humboldt's splendid work, entitled *Vues des Cordilleres*, exhibit the series of Mexican hieroglyphical numerals. To avoid the multiplication of the balls, and other

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symbols, those people sometimes divided the flag denoting a score by two cross lines, and coloured the one half of it to signify *ten*, or covered three quarters of it with colour to mark *fifteen*. This mode of abbreviating the signs was evidently capable of farther extension.

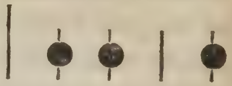
Traces of numeration by *scores* or *twenties* still exist in the old continent. The expression *three score* and *ten*, in our own language, is more venerable than *seventy*; and the compound *quatre-vingts et dix*, is the ordinary mode in French for signifying *ninety*. The inhabitants of the province of Biscay, and of Armorica, people descended from the ancient Celts, are said to reckon like the Mexicans, by the powers of *twenty*, or the terms of progressive *scores*.

The principle of numerical scales might likewise be conveniently employed in the exhibition of uniform systems of weights and measures. But, for this effect, as many distinct sets of ascending objects would be required, as one less than the number of units contained in the index of each progression. Thus, on the *Binary System*, *one* set of weights, rising by a successive duplication, is very generally used. If the *Denary Scale* were preferred, there would be wanted no fewer than *nine* sets of separate weights. This number might indeed be reduced to *four*, by employing in combination, besides the single series, others having double, triple, and quadruple its value.

Since, in the notation by numerical scales, the import of a counter depends on the position of the bar on which it stands, any alteration of the place of units must produce a proportional change on the value of the whole amount of an expression. Thus, in the *Binary* classification, the shifting of the bar of units one place lower, would, in effect, double all the preceding terms, and a second shift would double these again. In like manner, to carry the beginning of the scale a bar lower would, in the *Denary* system, convert the units into tens, the tens into hundreds, and the hundreds into thousands; thus augmenting the whole expression tenfold. On the contrary, if the units were moved to a bar higher, the amount of any expression would, in the *Binary* scale, be reduced to one-half, and, in the *Denary*, to the tenth part of its former value. Hence, to multiply or divide by the index of any scale or its powers, we have only to change the names of the bars, or to shift the place of the units.

The systems of progressive numeration are hence as well adapted to represent a descending as an ascending series; a property which greatly facilitates and simplifies the exhibition of fractions. Suppose, for example, it were sought to express *thirteen-sixtenths* on the *Binary Scale.* Since a counter depressed by *four* bars will signify only the *sixteenth* of its original value, so *thirteen*, reckoning upwards from the low bar, will express the value required.

BINARY SCALE.

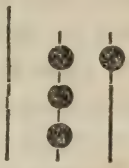


But the analysis of the fraction might be performed otherwise. It is evident, that *thirteen-sixteenths* on the *first* bar, or the bar of units, are only equivalent to *twenty-six* such parts, or *one* counter and *ten-sixteenths* placed on the descending bar. This excess again corresponds to *twenty*, carried to the *third* bar, making *one* counter and *four-sixteenths*. But these *four-sixteenths*, by successive duplication, pass over the fourth bar, and leave a whole counter on the fifth. The result is thus the same as before, and the fraction proposed appears to consist of *one-half*, *one-fourth*, and *one-sixteenth*.

On the *Quaternary Scale*, this fraction would require only *two* bars, since *sixteen* is only the *second* power of the index *four*.

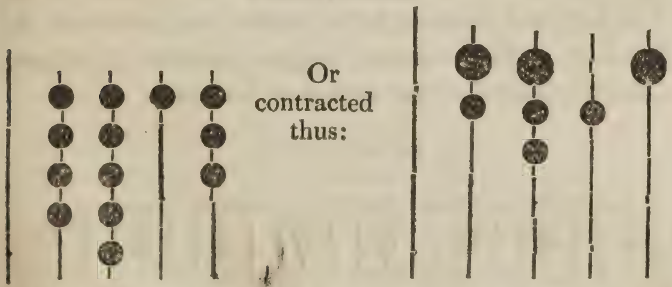
But the second mode of decomposition is, on the whole, simpler. The *thirteen-sixteenths* of a counter on the bar of units, are equivalent to *fifty-two*, or *three* counters, and *four* such parts on the next bar; and these *four-sixteenths* correspond to *sixteen*, which make a *whole counter* on the third bar.

QUATERNARY SCALE.



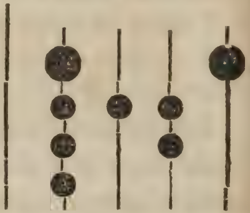
To express the fraction on the *Senary Scale*, more bars are wanted. It is now equivalent, on the bar immediately after that of units, to *seventy-eight-sixteenths*, or *four* counters and *fourteen* parts. This excess corresponds to *eighty-four*, or *five* counters and *four* parts, on the third bar. These *four* parts again give to the fourth bar *one* counter and *eight* parts, which correspond to *three* counters on the fifth bar.

SENARY SCALE.



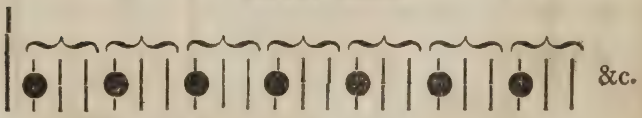

Let the same fraction be expressed by the process of successive decomposition on the *Denary Scale*. *Ten* times *thirteen-sixteenths*, or *one hundred and thirty* parts on the bar immediately below the units, make *eight* counters and *two* parts. These *two* parts are equivalent to *twenty* parts, or *one* bar and *four* parts on the third bar. This excess, again, gives *forty* parts, or *two* counters and *eight* parts to the fourth bar. And, lastly, these remaining *eight* parts are represented by *five* counters placed on the fifth bar. The fraction *thirteen-sixteenths* is thus analysed into *eight-tenths*, *one hundredth*, *two thousandths*, and *five ten thousandths*.

DENARY SCALE.



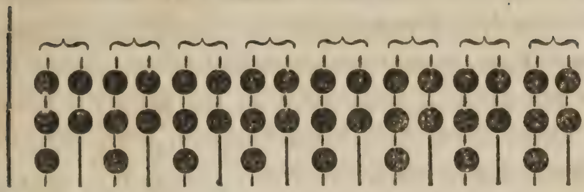
There is not the same facility, however, in decomposing all fractions, and reducing them to the terms of a descending scale. It often happens that the expressions for these will run through many bars, or even maintain a perpetual circulation, without ever drawing to a termination. Suppose, for example, that *three-sevenths* were to be represented on the *Binary Scale:* It would correspond to *six-sevenths* for the second bar; which is equivalent to *twelve* parts, or *one* counter and *five* parts on the third bar. These, again, make *one* counter and *three* parts for the fourth bar. This bar must therefore leave out the same portion as the first bar; and consequently the same notation will be continually repeated at the interval of three bars.

BINARY SCALE.



The same fraction would be thus expressed on the *Senary Scale:*

SENARY SCALE.

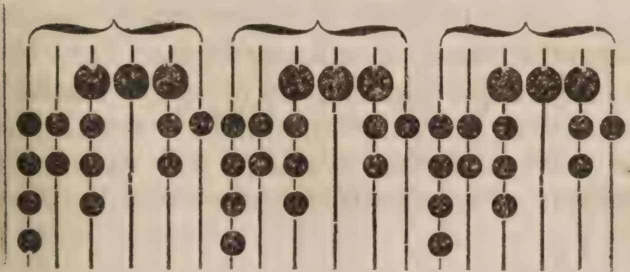


&c.

The *three-sevenths* of a counter are here equivalent to *eighteen* parts, or *two* counters and *four* parts on the second bar; and these *four* parts correspond to *twenty-four* parts, or *three* counters and *three* parts on the third bar. There being thus a remainder of *three-sevenths*, it is obvious that the process of decomposition will be incessantly renewed on the alternate bars.

If this fraction be reduced to the *Denary Scale*, the circulation will be still more complex. It will give for the second bar *thirty* parts, or *four* counters and *two* parts; for the third bar *twenty* parts, or *two* counters and *six* parts; for the fourth bar *sixty* parts, or *eight* counters and *four* parts; for the fifth bar *forty* parts, or *five* counters and *five* parts; for the sixth bar *fifty* parts, or *seven* counters and *one* part; and for the seventh bar *ten* parts, or *one* counter and *three* parts, which being the same remainder as at first, will occasion the perpetual recurrence of the same series of counters.

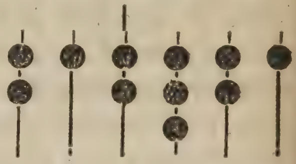
DENARY SCALE.



&c.

In a similar way, might compound fractions and numbers of involved denominations be reduced to arithmetical scales. Thus, to express *thirty-eight pounds sixteen shillings* and *tenpence halfpenny*, on the *Quaternary Scale*. The integer *thirty-eight* being analysed, gives *two* counters to the bar of units, *one* to the next higher bar, and *two* to the bar above this. Again, the fractional parts of a pound, if quadrupled and placed on the bar below that of units, makes *three* pounds *seven* shillings and *six* pence ; this excess, again, carried to the next lower bar, is augmented to *two* pounds and *five* shillings ; and these *five* shillings are equivalent to *one* pound on the third descending bar.

QUATERNARY SCALE.

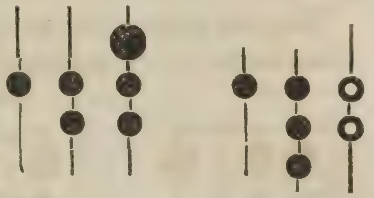


But it seldom happens that the expression of such complex quantities terminate so soon, or indeed close without circulation. To reduce a fraction to any descending scale, may therefore prove a tedious, and often impossible task. But the converted expression approximates rapidly to its true value, and a very few terms will be sufficient for every practical use.

The numerical scales are thus equally fitted by their constitution for ascending or descending,—whether for exhibiting huge multitudes or the minutest subdivision of parts. But men were, in general, very slow to perceive or to avail themselves of this most valuable though distinguishing property of such progressions. The Chinese are the only people who have for ages been accustomed to employ the descending terms of the *Denary Scale*, or to reckon by *Decimal* parts in all their commercial transactions. The same uniform system directs the whole subdivision of their weights and measures; an advantage of the highest importance, since it gives to the calculations of those ingenious traders the utmost degree of simplicity and readiness.

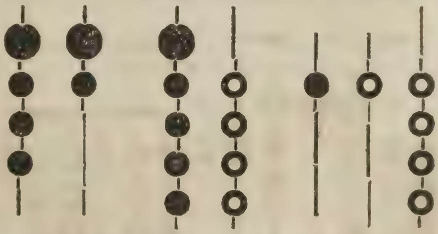
When the index of a *Numerical Scale* is large, the notation may be conveniently abridged, by marking only what counters are wanted to complete any bar, or render its expression equivalent to that of an additional counter placed on the bar immediately before it. Thus, instead of *eight* counters on a particular bar, it would be the same thing, to join *one* to the preceding bar, and put *two* deficient or open counters in the *Denary Scale*, or *four* such counters in the *Duodenary Scale*. For the sake of illustration, let some of the former examples be resumed. The number *eighty-six*, as represented on the *Octary Scale*, may have the *six* counters on the bar of units changed, by substituting *two* deficient counters, and joining *another* counter to the bar immediately above it, as here exhibited.

I. II.



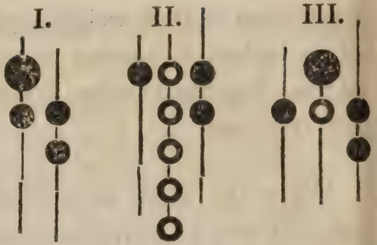
On the *Denary Scale*, the expression for the same number will thus appear successively changed.

I. II. III.

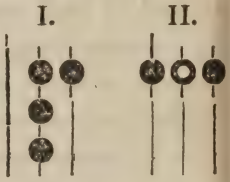


The last form of notation, consequently, signifies *one hundred*, abating *fourteen*.

Again, the number *eighty-six*, as represented on the *Duodenary Scale*, may be thus modified; implying *a gross* and *two*, diminished by *five* dozen.

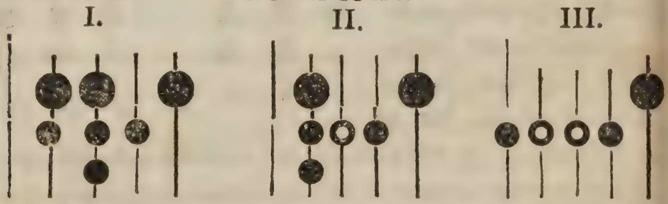



This method of employing open or deficient counters is applicable likewise to the notation of fractions. Thus, the fraction *thirteen-sixteenths* may be expressed two different ways on the *Quaternary Scale*; the second form intimating it to be the same as *one* and a *sixteenth* diminished by a *fourth*.



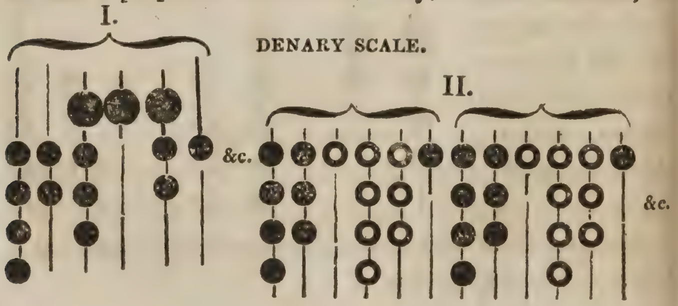

The same fraction may be represented in three distinct forms on the

SENARY SCALE.




The fraction *three-sevenths*, which, on the *Denary Scale*, formed a perpetual recurrence, may, in like manner, be

DENARY SCALE.



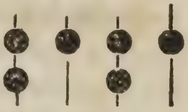
abbreviated. The first expression is converted into another more commodious one, by changing the counters that exceed *five* on any bar into deficient counters. There is also the same circulation as before, at the interval of *six* bars.

From the application of the same principles, it is easy to reduce any number expressed by rows of counters, to its original heap. Thus, in the annexed arrangement on the *Binary Scale*, the counter on the highest bar is equivalent to *two* on the next bar, to *four* on the bar below this; and so forth by a repeated duplication, till it leaves *sixty-four* in the bar of units. In like manner, the counters on the third and second bars, carried downwards, give *four and two* to the bar of units; and consequently the aggregate amount of this decomposition is *seventy one.*

BINARY SCALE.

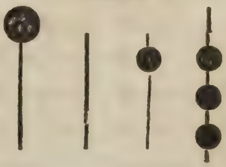
In the *Ternary Scale*, the corresponding expression is thus marked. But the *two* counters on the highest bar correspond to *six* on the next, to *eighteen* on the following bar, and to *fifty-four* on that of units. The *single* counter on the *third* bar is equivalent to *nine* on the bar of units; the *two* lower counters are equivalent to *six* on the same bar, which therefore holds *two*, *six*, *nine*, and *fifty-four*, amounting in all to *seventy-one* units.

TERNARY SCALE.

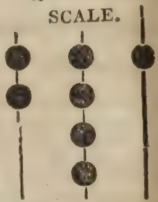


In the *Quaternary Scale*, the *fourth* and *second* counters correspond to *sixty-four* and *four*, on the bar of units; which, with the *three* counters already occupying that bar, make up *seventy-one.*

QUATERNARY SCALE.



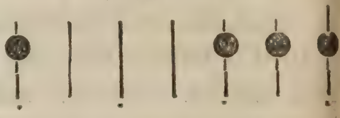
QUINARY SCALE.



Lastly, in the *Quinary Scale*, the *two* counters on the third bar would give *ten* to the next, which joined to the *four* counters placed on it, would furnish *seventy*, to be annexed to the *one* occupying the bar of units.

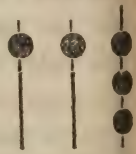
Numbers thus reduced to their primary elements, might be distributed again into other classes, and consequently arranged on any scale. But in certain cases, they can be transferred directly to a higher scale, without undergoing such previous decomposition. This will happen, whenever some power of the one index becomes equal to a power of the other, or when the result of their repeated multiplication actually coincides.

BINARY SCALE.



Thus, resuming the expression of the *Binary Scale*, if the alternate bars were omitted, and their values cast to the bars immediately below them, which are marked here with dots, the whole would be converted into the *Quaternary Scale*. The counter on the *second* bar furnishes *two* to that of units; the counter on the *third* bar remains untouched; the *fifth* bar remains vacant, and forms the *third* in the new arrangement; and the counter on the *seventh* bar continues unaltered.

QUATERNARY SCALE.



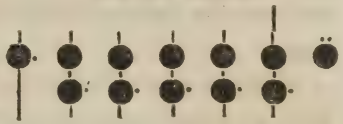
Again, by condensing three bars into one, the expression of the *Binary Scale* will be changed into another on the *Octary Scale*. Here the *fourth* bar transfers its vacuity to the *second*, the *seventh* bar is converted into a *third*, while the counters which occupied the *third* and *second* bars increase the number of units to *seven*.

OCTARY SCALE.

In like manner might any expression on the *Ternary*, be reduced to one on the *Nonary, Scale*, by condensing every pair of bars into a single bar. On the same principle, the exhibition of a number by the *Quaternary Scale* would be transferred to the *Octary Scale*, by reducing the import of *three* bars always to that of *two* bars, since *four* multiplied *three* times gives the same result as *eight* multiplied *twice*.

We shall now investigate some general properties of those *Numerical Scales*. Suppose, in the *Binary Progression*, there was standing but a single counter on a high bar. It is obvious, that this counter might be removed, and *two* such placed on the inferior bar. But *one* of these might likewise be removed, and *two* counters substituted for it on the bar next lower. The same process could be pursued through any number of bars, the removed counters being always marked by a dot, and the one which is finally rejected placed on the outside of the last bar with two dots over it.

BINARY SCALE.



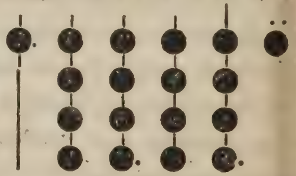
It hence appears, that *one* placed on any bar of the *Binary Scale*, is equal in value to *one* joined to the sum of a series of units running through all the inferior bars. Thus, in the example now produced, *one* counter occupying the *sixth* bar, and therefore indicating the number *thirty-two*, is equal in effect to *one* annexed to the sum of *sixteen, eight, four, two*, and *one*.

Let similar modifications be introduced into the *Ternary Scale*. Suppose a single counter to stand by itself, all the rest of the bars being empty. It may be taken

TERNARY SCALE.

away, and *three* counters substituted for it on the next inferior bar. But *one* of these may be now withdrawn, and *three* others placed on the following bar: Of this triplet, the undermost might again be removed, and *three* substituted for it on the next bar. The same process, it is evident, could be repeated, till the change reached the lowest bar, leaving out an excess of *one*, marked by two dots to signify its being transferred.—Hence the single counter on the *Ternary Scale* is, by successive mutations, converted into *two* rows of counters extending through all the inferior counters, and leaving an excess of *one*. Thus, the number *two hundred* and *forty-three*, the value of a counter placed on the *fifth* bar of the *Ternary Scale*, is equal to *one* added to double the sum of *eighty-one*, *twenty-seven*, *nine*, *three*, and *one*, the values of counters occupying all the inferior bars.

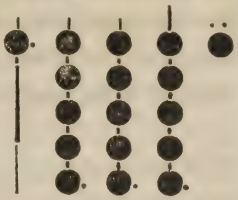
QUATERNARY SCALE.



In like manner, if a solitary counter in the *Quaternary Scale* be withdrawn, *four* counters may be substituted on the next bar. Remove the undermost of these, and set *four* more on the succeeding bar. Take away *one* of these again, and put other *four* counters on the adjacent bar. Proceed in the same way, till the quaternion reaches the last bar, and is reduced to a triplet, by the exclusion of *one* counter.—By this analysis, therefore, the simple counter is resolved, with an excess of *one*, into *three* rows of counters, which run through the whole of the lower bars. In the present instance, the number *two hundred and fifty-six*, the import of a counter on the *fifth* bar of the *Quaternary Scale*, is equivalent to *one* joined to *triple* the sum of *sixty-four*, *sixteen*, *four*, and *one*, the values of all the succeeding bars.

It may seem scarcely necessary to pursue this investigation farther, but we shall extend it likewise to the *Quinary Scale*. A single counter, it is obvious, may now be removed, and *five* substituted for it on the next bar. The undermost of these, again, may be withdrawn, and *five* placed instead of it on the following bar. One of these may then be taken away, and *five* substituted for it on the adjacent bar. The same procedure is repeated to the last bar, leaving *four* rows, with an excess of *one* counter.

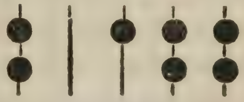
QUINARY SCALE.



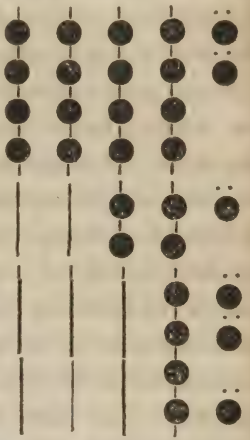
In the present instance, a single counter on the *fourth* bar, and corresponding, therefore, on this scale to the number *six hundred and twenty-five*, is equivalent to *one* added to *four* times the sum of *one hundred and twenty-five*, *twenty-five*, *five*, and *one*, the values of all the inferior bars.

We may hence conclude in general, *that if one be taken from any power of the index of any numerical scale, the remainder will be equal to all the inferior powers repeated as often as the units in that index diminished by one.* Wherefore if the counters, reckoned as mere units, be separated from any compound expression, the whole will be converted into as many trains of counters, occupying all the inferior bars, as correspond to the index of the scale diminished by *one*. Suppose, for example, the expression here noted, which is disposed on *five* bars of the *Ternary Scale*, and is equivalent to *one hundred*

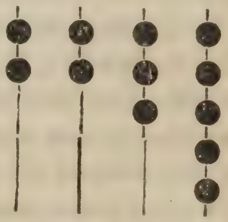
TERNARY SCALE.

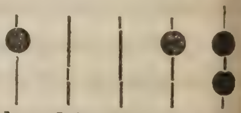


*and seventy-eight.* By decomposing separately each successive bar, and placing the excluded counters close beside the place of units, it will be changed into this regular but complex form, which consists of *two* rows of *two* counters, leaving out *two*; *two* rows of *single* counters, leaving out *one*; the *two* counters on the last bar but one *redoubled*, the final counter being excluded.



If we omit altogether the *six* excluded counters, and take only the single rows of the rest, the result must evidently express the quotient of the remainder by *two*, or by the index of the scale diminished by *one*. Collecting these counters on the *Ternary Scale* into a more compact form, the expression below is obtained; which corresponds to *eighty-six*, or the half of *one hundred and seventy-two*, the original number, leaving out the *six* counters employed in denoting it.

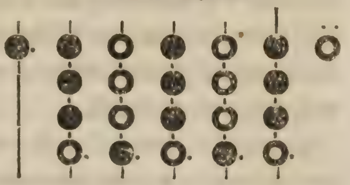




Another similar property, belonging to numerical scales, may be deduced from the combination of the deficient or open counters. Let us begin with the *Binary System.* Suppose a solitary counter to occupy the *sixth* bar. It may be removed, if *two* counters be placed in its stead on the next bar. But, without changing their value, we may to this pair evidently join another, composed of a full and an open counter, which perfectly balance each othe

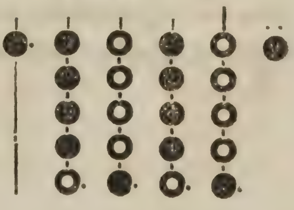
Withdraw the open counter that stands undermost, and substitute for it *two open* counters on the fourth bar. To these again, add a balanced pair, consisting of an open and a full counter. Take away the undermost counter, set *two* similar counters on the third bar, and to this pair annex a full and an open counter. By continuing this process, the single counter will be decomposed into *three* rows of counters, alternately full and open, with an excluded counter, which is open when the number of the bars, as in the present case, is even, but full if that number be odd.—Thus, *thirty-two*, the value of the counter on the sixth bar, is equal to *three* times all the inferior alternating bars, or the excess of *sixteen* above *eight*, joined to the excess of *four* above *two*, together with *one*, that is, *eleven*; abating, however, the excluded counter which is here open; but thrice *eleven*, omitting *one*, is *thirty-two*.

BINARY SCALE.



Let a similar analysis be applied to the *Ternary Scale*. Change the solitary counter for *three* counters laid on the next bar, and to these join a balanced pair, consisting of a full and an open counter. Remove this open counter, and substitute *three* open counters for it on the succeeding bar, and to the triplet annex an open and a full counter. Take away the full counter, and place *three* such counters on the following bar. Repeat the procedure, till the first bar comes to be occupied; and there will evidently emerge *four* rows of counters extending through all the inferior bars, and alternately full and open, with an excluded counter of

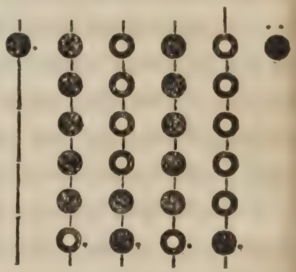
TERNARY SCALE.



an opposite character to those which terminate the decomposition. In the present instance, where the single counter stood on the *fifth* bar, a counter of the ordinary kind is left out. Wherefore the number *eighty-one* is equal to four times the amount of the excess of *twenty-seven* above *nine*, and of that of *three* above *one*, or *twenty*, together with the *unit* excluded.

It may be deemed sufficient for grounding a general conclusion, to repeat the same process on the *Quaternary Scale*. Instead of the solitary counter, place *four* counters on the next bar, and conjoin with these a balanced pair of counters, composed of a full and an open one. Remove the undermost of these, and substitute four open counters on the following bar.

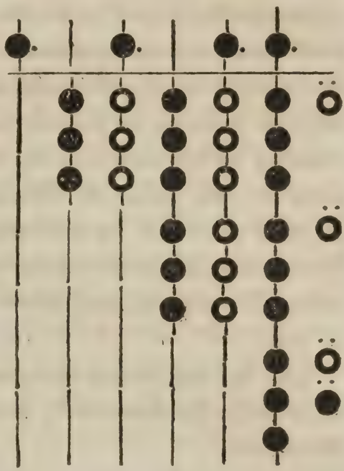
QUATERNARY SCALE.



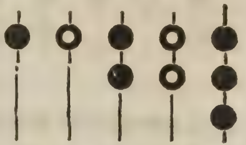
To these, again, add an open and a full counter, which will not affect the value of the column. Pursue the operation till all the bars are occupied by counters, and *one* excluded. It is evident therefore, that, on the *Quaternary Scale*, the decomposition of a single counter produces *five* rows of alternating counters through all the inferior bars, with an excluded counter of an opposite nature to that of the row which completes the analysis. In the example now given, the number *two hundred and fifty-six*, the value of the counter on the *fifth* bar, is equal to five times the amount of the excess of *sixty-four* above *sixteen*, and of *four* above *one*, or *fifty-one*, that is, *two hundred and fifty-five*, together with the *unit* left out.

The conclusion may, therefore, be extended to any progressive scale: *If the value of unit on a separate bar be increased or diminished by one, according as the rank is even or odd, the remainder will be divisible by one greater than the index of the scale, and the quotient will be equal to the amount of the values of unit, as alternating in excess and defect through all the inferior bars.*

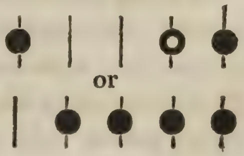
BINARY SCALE.



The same property, it is evident, could be transferred to any compound expression: Nothing is wanted but to change the counters on each successive bar into alternating rows.—Thus, in the *Binary Scale*, the expression which signifies *forty-three* will be converted into another composed of *triple alternating rows*, with an exclusion of *one* counter in excess, and *three* in defect, or the balance of *two* open counters. If this arrangement be now divided by *three*, the result, after restoring the deficient unit, will be as here exhibited.

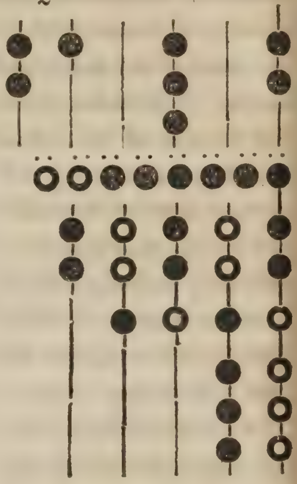


By collecting and condensing the counters on each bar, the whole will be reduced to another very simple form; which indicates *fifteen*, the third part of *forty-five*, or of the original number, and *two* counters more.



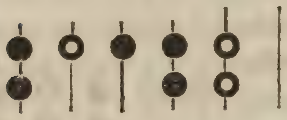
Let another example be selected from the notation of the *Quaternary Scale.* The counters on these six bars will represent the number *two thousand three hundred and fifty-four.* The excluded counters will range as here noted; and, consequently, there is on the whole an excess of *four* counters which must be rejected.—The original number is, therefore, reduced to *two thousand three hundred and fifty,* of which the *fifth* part will be denoted by repeating the counters alternately full and open on the lower bars.

QUATERNARY SCALE.

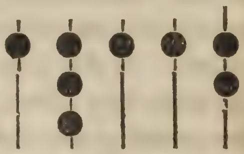


This last expression being condensed and abridged, will stand as in the first form; or with full counters only, as in the second.

I.



II.



The amount of the division by *five* is evidently *four hundred and seventy.*

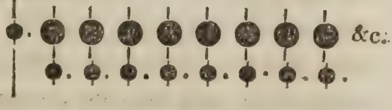
We may hence conclude generally, that *if a number expressed on any scale be diminished by the counters on the alternate bars beginning with that of units, and augmented by the counters on the even bars, the result will be divisible by one greater than the index, and the quotient will be represented by repeating the counters through all the descending bars of alternate character.*

In all these decompositions, we have stopped at the bar of units; but if we pursue the analysis through the descending bars, we shall discover trains of equivalent fractions which never terminate. Thus, to begin with the *Binary Scale:* A counter on the bar of units may be taken away, and two counters placed instead of it on the following bar. Of this pair, again, one may be removed, and another pair substituted for it on the next lower bar. One of these, again, may be withdrawn, and two placed on the following bar. The same operation of exchange, it is obvious, may be repeated for ever. Wherefore, the value of a single counter is here the same as that of a single row of counters, extending indefinitely over the lower bars. But the counter on the bar immediately below the place of units, indicates *one-half*, that on the next *one-fourth*, that on the following bar *one-eighth*, and so forth continually. Wherefore the sum of the fractions *one-half*, *one-fourth*, *one-eighth*, *one-sixteenth*, extended without limit, must always approach to *unit* or *one whole*.

BINARY SCALE.


Let a similar transformation be carried through the *Ternary Scale*. Suppose a half counter to stand on the bar of units: It may be removed, and *three* half counters, or a whole counter and half of one substituted on the next bar. Take away this half counter, and set *three* such, or a counter and a half, on the succeeding bar. Repeat the same process continually, and the half counter on

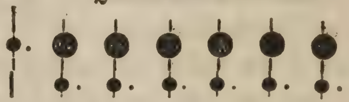
TERNARY SCALE.



the bar of units will be converted into a single row of entire counters, extending without limitation through all the inferior bars. But these successive counters signify *one-third*, *one-ninth*, *one-twenty-seventh*, &c. Whence the fraction *one-half* is equal to the sum of *one-third*, *one-ninth*, *one-twenty-seventh*, &c. continued without end.

In the *Quaternary Scale*, let the *third* of a counter occupy alone the bar of units. It may be withdrawn, and *four* such parts, or a whole counter, and the *third* of one placed in its stead on the next bar.

QUATERNARY SCALE.

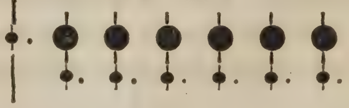


&c.

This *third*, again, may be removed, and a counter, with another *third*, substituted for it on the following bar. The same procedure being repeated, the *third* part of a counter in the place of units will be changed into a row of entire counters running through all the inferior bars. It therefore follows, that the fraction *one-third* is equal to the sum of the infinite series *one-fourth*, *one-sixteenth*, *one-sixty-fourth*, &c.

Again, let similar modifications be carried through the *Quinary Scale*. The fourth of a counter on the bar of units may be exchanged for *five* such parts, or one counter and a quarter on the following bar; and this quarter may now be removed, and *five quarters*, or *one* counter and a *quarter* set on the next bar. The process of decomposition may thus be continued perpetually, leaving instead of the *fourth* of a counter, an unlimited range of counters stretching over the inferior

QUINARY SCALE.



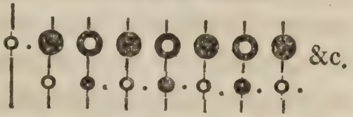
&c.

bars. Consequently the fraction *one-fourth* is equal to the aggregate terms of the progression *one-fifth*, *one twenty-fifth*, *one hundred and twenty-fifth*, *one six hundred and twenty-fifth*, continued without termination.

From these very simple analyses, we may therefore conclude in general, *that the fraction of unit, which has for its denominator one less than the index of any numerical scale, is equal to the sum of all the descending powers, or the value of a single row of counters, extending indefinitely through the inferior bars.*—Thus, *one-ninth* is equal to a *tenth*, a *hundredth*, a *thousandth*, &c. or *one-eleventh* is equal to a *twelfth* part, a *hundred* and *forty-fourth*, a *thousand seven hundred and twenty-eighth*, &c.

But the summation of a descending series, whose terms alternate, may with equal facility be discovered, by introducing the admixture of deficient counters.—Thus, not to multiply examples, suppose on the *Ternary Scale* the *fourth* part of a counter occupied the bar of units. Remove this, and substitute *three-quarters*, or a whole counter abating *one-quarter*, on the next bar. Instead of this *deficient quarter* again, place *three* such, or *one open* counter, conjoined with the *quarter of a full counter*, on the succeeding bar. Repeat the same procedure, and the quarter of a counter will be transformed into a single row of counters, alternately full and open, extending without limitation over the lower bars. Wherefore the fraction *one-fourth* is equal to the excess of the perpetual series *one-third*, *one twenty-seventh*, &c. above the similar series, *one-ninth*, *one eighty-first*, &c.

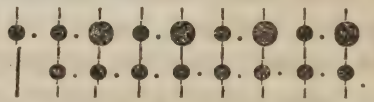
TERNARY SCALE.



We may hence infer generally, *that the fraction of unit, divided by one greater than the index of any numerical scale, is equal to the amount of all the descending powers taken alternatively as additive and subtractive.*

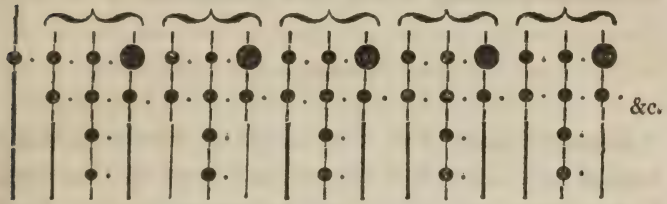
In all these transformations of fractions, arising from the index of the numerical scale, increased or diminished by *one*, the operation is repeated or alternated at each successive bar. But similar changes may be made on fractions derived from the same modifications of the powers of the index, which will regularly circulate along the bars at a corresponding interval. Thus, on the *Binary Scale*, the fraction *one-third*, or the second power of the index two diminished by one, will form by decomposition an intermitting row, or a perpetual circulation, passing over the successive alternate bars. For *one-third* of a counter on the bar of units is equivalent to *two-thirds* on the following bar, which again are equal to *four-thirds*, or an entire counter and a *third*, on the next bar. Pursuing the same analysis, a row of counters emerges on the alternate bars. In reality, if the intermediate bars, which here serve only for the transit of the pair of *thirds*, were left out altogether, the notation would pass into that of the *Quaternary Scale*, and obey the general rule.

BINARY SCALE.




Again, on the same *Binary Scale*, the fraction *one-seventh*, or the reciprocal of the third power of two, diminished by *one*, will be found to circulate at every *third* bar. Thus, *one-seventh* of a counter on the bar of units gives *two* such parts for the second bar, *four* for the *third* and *eighth*, or a whole counter and an excess of *one-seventh* for the *fourth* counter; and if this kind of decomposition be carried forward, another counter will appear on the *seventh* bar, a *third* on the *tenth* bar; and so forth in perpetual succession.

BINARY SCALE.




But the same conclusion might also be drawn from the general principle, if we consider that the *Binary Progression*, by omitting always two consecutive bars, is converted into the *Octary Scale*.

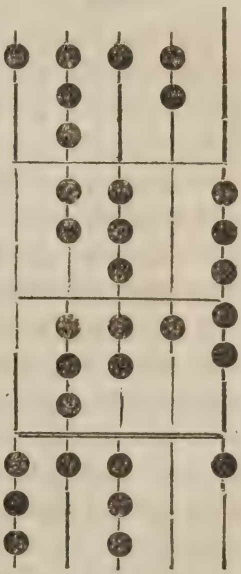
It is not difficult to perceive, that every fraction is capable of being either exactly represented on any given scale, or of being denoted by an expression which circulates after an interval of fewer bars than the denominator of the fraction contains units. In fact, the moment the same set of fractional counters comes to appear a second time, the whole expression must evidently recur in the same order. But all the possible variations or series of remainders must ever lie within the number itself, which constitutes the divisor. Thus, it was found that the expression for any fraction having the denominator *seven*, circulates on the *Binary Scale*, at the interval of *three* bars. The same fraction represented on the *Quaternary Scale* has a like recurrence; but, on the *Octary Scale*, the expression is renewed at every *two* bars, while it does not circulate till after passing over *six* bars, in the *Ternary, Quaternary, Quinary,* and *Denary Scales.* Employing a similar decomposition, it will appear that a fraction, with *eleven* for its denominator, will, in the *Quaternary* and *Quinary Scales*, circulate on *five* bars, but will embrace no fewer than *ten* bars, by its circulation in the *Ternary, Quaternary, Senary, Octary,* and *Denary Scales.*

Having considered at some length the properties of numerical scales, and their various transformations, we have now to explain the ordinary operations performed on numbers themselves. These operations are all reducible to two very simple changes,—the *conjoining* and the *separating* of numbers. When two or more numerical expressions are *conjoined*, that is, condensed into a single expression, or collected into one *sum*, the process is called *Addition.* But when one numerical expression is *separated* or drawn out from another, leaving only a *difference* or *remainder*, the process is called *Subtraction.* If the *addition* should be employed merely in repeating the same number, it admits of abbreviation, and is then termed *Multiplication.* On the contrary, if the *subtraction* be limited to the continued withdrawing of the same number from another, the process becomes capable of abridgment, and is termed *Division.*

## ADDITION.

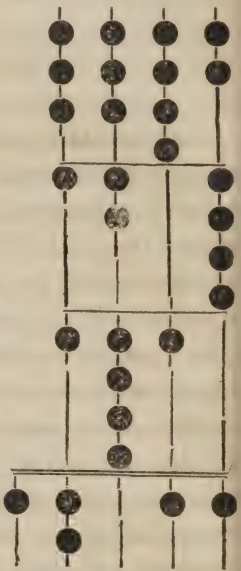
The whole operation consists in collecting and condensing the separate expressions. Beginning with the lowest bar, the counters are gathered together, and if they exceed the index of the scale, this excess only is retained, and *one* counter annexed or carried to the next bar. But if the counters on any bar should contain the index more than once, the number of repetitions is transferred a place higher, while the remainder of the reckoning is left as it stood. A very few examples will render the mode of proceeding quite clear. Let it be required to collect the expressions here disposed on the *Quaternary Scale*, and corresponding to *four hundred and seventy-two*, *one hundred and seventy-nine*, *and two hundred and thirty*. On the bar of units, *five* counters occur, which leave *one*, and advance *one* to represent *four* on the next bar. This second bar now holds *four* counters, which, therefore, leave it vacant, and furnish *one* to represent them on the third bar. On this, again, *seven* counters are found, leaving *three* consequently, and furnishing *a counter* to the fourth bar, which thus comes to contain *nine* counters. *One* counter is, therefore, left, and *two* carried to the counter occupying the highest bar. The sum of those three compound numbers hence corresponds to *eight hundred and eighty-one.*

QUATERNARY SCALE.

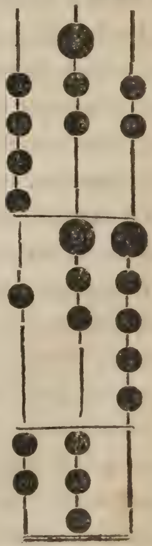


Let the same numbers be transferred to the *Quinary Scale.* On the bar of units, *six* counters occur, which leave an excess of *one*, and send another to represent *five* on the next bar. This second bar has now likewise *six* counters, and hence leaves *one*, and advances *one* to the higher bar. The third bar, containing *ten* counters, is left vacant, after giving two representatives to the *fourth* bar. This last bar now holds *seven* counters, or retains *two*, and sends *one* to the highest bar. The expression for the sum corresponds, as before, to *eight hundred and eighty-one.*

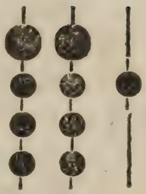
QUINARY SCALE.



DENARY SCALE.

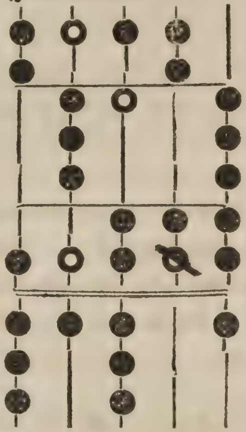


Let those numbers be arranged on the *Denary Scale;* they will be thus represented. In the bar of units, the *four* lower counters, with one of the upper, leave *a counter*, and furnish *another* to represent the two five on the next bar. This advanced counter, joined to the single counters on the second bar, make *five*, with an excess of *three*, while the two remaining fives give a counter to the higher bar, making likewise *five* counters, with an excess of *three.* The counters on the several bars being now collected on the opposite page, ive obviously the same result as before.



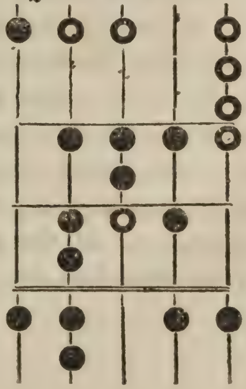
The working of these examples would be generally simplified by he judicious application of deficient counters. Thus, in the first expresion of the *Quaternary Scale*, *one* counter being on the highest bar, *our* counters may be taken up, or n *open one* left. The same process nay be applied to the two following xpressions; and farther, the last one nd additional counter being placed on the second bar, *four* couners may be taken up from the bar f units, leaving *two* open counters. Collecting, therefore, the counters n the several bars, and observing he opposite effects of the full and the open, there is a baance of *one* counter on the first bar; the *four* counters on he second bar leave it vacant, and throw one to the next, n which the partial balance gives an excess of *three*; the pposite counters on the fourth bar leave a balance of *one*; nd *three* counters are still found on the highest bar.

QUATERNARY SCALE.

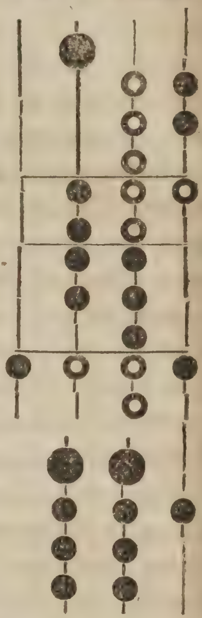


If open or deficient counters be dopted in the expressions on the *Quinary Scale*, the quantities will e thus denoted. The *four* deficient counters on the bar of units re equivalent to *one* full counter n the same bar, and one taken from he next bar. In the third bar, he opposite counters are exactly balanced, and in the fourth bar there is an excess of *two* counters.

QUINARY SCALE.



DENARY SCALE.

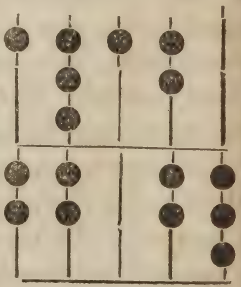


Let the numbers be expressed by open counters on the *Denary Scale.* The arrangement will stand thus: But, on the bar of units, setting aside the full and the deficient counter, *one* full counter is left; on the next bar, the *three* full balance *three* of the open counters, leaving another open *one*; on the third bar the *four* full counters are equivalent to *five* such, with an open one.

The whole amount may be expressed otherwise thus: there being *eight* counters on the second and on the third bars, and *one* counter on the first.

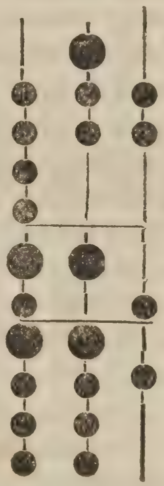
It is more natural, however, and more like the spontaneous practice of uninstructed men, to proceed by successive steps, and only conjoin two numbers at the same time. Nor is it requisite, in this mode of proceeding, that the numbers to be severally added be actually expressed; it may be sufficient, at each advance, to retain them mentally. Thus, on the *Quaternary Scale*, the first number being represented, *three* counters belonging to the second number are to be laid on the first bar, *none* joined to those of the second bar, *three* counters added to the *one* on the third bar, leaving it consequently vacant, and throwing *one* to the fourth bar. On this fourth bar, there now occur

QUATERNARY SCALE.



*four* counters, which furnish *one* counter to the fifth bar, and leave the *two* annexed counters. Again, the third number gives *two* counters to the bar of units, making up *five*, or leaving *one* counter, and carrying *another* to the next bar, on which the transferred counter, with *one* to be joined to it, make *four*, and consequently leaving a void, send a counter to the third bar. The third bar having an accession of *two* counters, now holds *three*, while the *two* counters of the fourth bar, joined to other *three*, give *five* or leave a counter, and furnish one to augment those of the fifth bar to *three*.

DENARY SCALE.

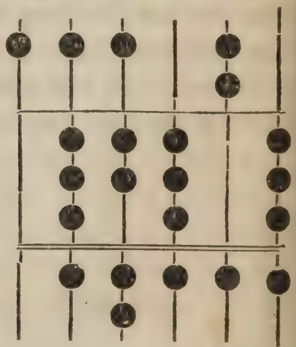


Let the same process be performed on the *Denary Scale*. To the *two* counters on the first bar, *nine* being joined, give eleven, or an excess of *one*, and another carried to the next bar, which, by the accession of *seven* more, leaves *five* counters, and sends another counter to the third bar, where *six* are now collected. The third number does not affect the bar of units; it furnishes *three* counters, however, to the *five* of the next bar, and *two* more to the *six* of the third bar.

## SUBTRACTION

Is that process by which a number is severed or drawn out from another. The number so parted is called the *Minuend;* the one detached from it the *Subtrahend;* and what is left after of the separation forms the *Remainder* or *Difference.* If the counters representing the minuend exceed on each bar those denoting the subtrahend, we need only mark the several excesses. But if the minuend have fewer counters on any bar than the subtrahend, it will be necessary to carry the decomposition farther, by taking a counter from the next higher bar, and throwing its value to the expression of the other, by joining as many counters as there are units in the index of the scale. Suppose it were required to subtract *nine hundred and forty-seven* from *thirteen hundred and fifty-two,* as thus arranged on the *Quaternary Scale.* Here the object is, without disturbing the order of the counters, to tell out from those of the minuend the expression of the subtrahend, and note the residue. As no counter appears on the first bar of the minuend, a counter is taken from the second, which, having the value *four,* gives the three counters of the subtrahend, and an excess of one. On the second bar there is nothing to take away, and consequently the single counter now left is placed below.

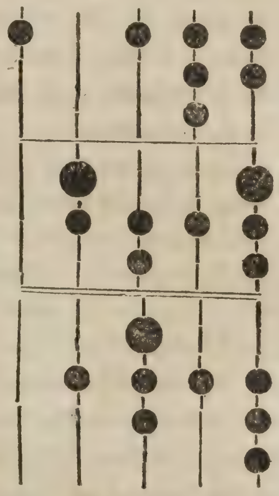
QUATERNARY SCALE.



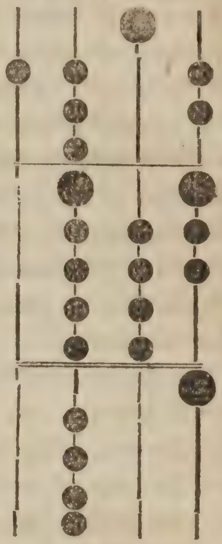
To supply the vacancy of the third bar of the minuend, a counter is borrowed from the fourth, and its value represents the *three* counters of the subtrahend, with a surplus of *one.* The fourth bar is likewise augmented by four, by anticipating the counter of the next bar, and gives an excess of *two.* By taking the highest counter again, a difference of *one* is left on the fifth bar. The whole remainder expresses *four hundred and five;* and, as in the process of subtraction, the minuend was only split into two portions, so these combined again must form the original number.

Let the same operation be repeated on the *Senary Scale.* In this case the *two* counters of the minuend, in the bar of units, are, by help of one borrowed from the higher bar, augmented to *eight*, which gives *five* for the subtrahend, and leaves an excess of *three.* On the next bar, the *two* counters now left furnish a counter to the subtrahend, and another to the remainder. The counter on the third bar of the minuend, increased by six, the value of one that should be drawn from the fourth bar, deposits *two* on the subtrahend, and delivers over *five* to the remainder. The counter borrowed from the fourth bar is supplied from the six counters corresponding to the expression of the highest bar; so that *four* counters are dropped, and *one* furnished to the remainder, which has collectively the same value as before.

SENARY SCALE.



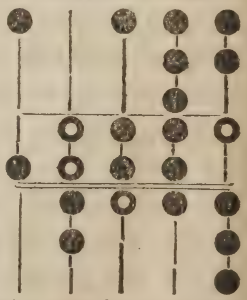
DENARY SCALE.



Lastly, suppose this subtraction were performed on the *Denary Scale.* The *two* counters on the bar of units being augmented to *twelve* by the counter borrowed from that of tens, give *seven* to the minuend, with an excess of *five.* The *five* counters on the bar of tens, reduced now to *four*, merely supply the subtrahend, leaving the remainder vacant. On the bar of hundreds, the *three* counters increased by the value of the higher counter, furnish *nine* to the subtrahend, and leave *four* for the remainder.

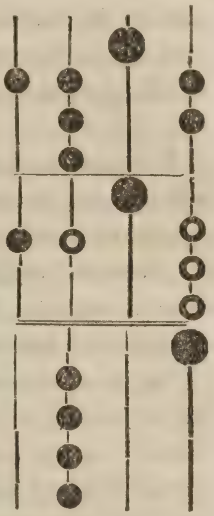
These operations may sometimes be conveniently abridged, by the judicious introduction of open counters; the value of the last bar will not be altered. Thus, on the *Senary Scale*, if a full and an open counter be annexed; consequently, while the open counter of the subtrahend is supplied, *three* full counters are thrown to the remainder. On the second bar the *three* counters give *two* to the subtrahend, and *one* to the remainder. The single counter on the third bar, being combined with a full and an open counter, supply the *two* counters below, and surrender this excess of an open counter. On the fourth bar, the vacuity may be occupied

SENARY SCALE.



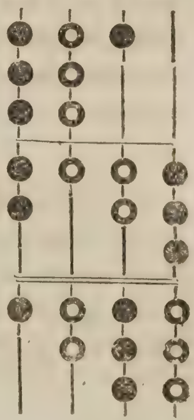
by *two* full and *two* open counters, and consequently the latter go to the remainder.

DENARY SCALE.



On the *Denary Scale*, the process of subtraction is likewise shortened. The *two* counters of the bar of units having three full and three open counters annexed, furnish the latter to the subtrahend, and give five full counters to the remainder. On the bar of tens, the counters of the minuend and of the subtrahend are equal, and consequently leave a vacuity. But on the bar of hundreds, the three counters with a full and an open combined, surrender the latter to the subtrahend, and deliver four full counters to the remainder.

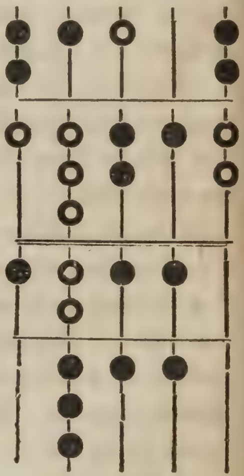
In every instance where a counter is borrowed from a higher bar, the effect would evidently be unaltered, if a counter were added on the same bar to the number below. This modification of the process is what has been generally termed *carrying*. It is farther illustrated, by operating with open counters. Thus, resuming the foregoing example, which might be expressed in this way on the *Octary Scale*. Conceive three full and three open counters were placed on the first bar, and the latter would evidently go to the remainder. Again, on the second bar, two open, and two full counters, would



throw three full counters to the remainder. On the third bar, two of the open counters are left in excess; and on the fourth bar, there is an excess of a full counter.—The result may be changed, as here done, into a more commodious form.

It is obvious from the procedure now followed, that the effect would be exactly the same, if the counters of the subtrahend, converted into others of an opposite character, were all conjoined with those of the minuend. By such a change, the operation of subtraction is readily transformed into one of addition.

To illustrate this remark, we may take any of the former examples. Suppose the expressions on the *Quinary Scale* to be assumed, but the counters of the subtrahend inverted. The various counters on the several bars being now collected and balanced, would give the annexed result; which, being modified again, forms the remainder of the original subtraction.



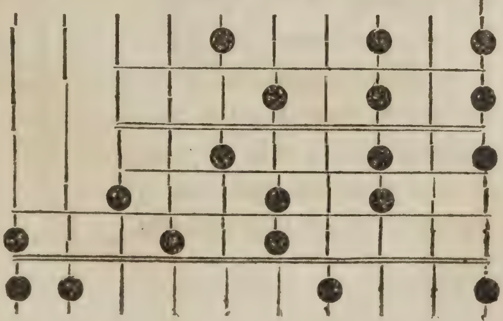
In all these operations, the procedure is alike, whether on the ascending or the descending bars. Hence fractions may be added or subtracted by help of counters, precisely in the same way as integers themselves. It would be superfluous, therefore, to produce any examples.

## MULTIPLICATION

Is nothing but a process of repeated Addition. When the terms, however, to be multiplied are complex, and the index of the *Numerical Scale* is large, the operation will admit of being very considerably abridged. It has been already shown, that a number is virtually multiplied by the *index* of the scale, by advancing its expression *one* bar; that it is multiplied by the *second power* of that index by advancing it *two* bars; and so forth continually, according to the progressive powers. Again, if any term of the multiplier be great, it is preferable, instead of repeating the counters of the multiplicand, to collect them mentally, and only to mark the result. The ready performance of multiplication depends entirely on the right application of these two principles. A few examples will elucidate the process.

Suppose it were required to multiply the number *thirty-seven* by *twenty-one*, that is, to add *twenty-one times* together the units contained in *thirty-seven.* First, let those numbers be disposed on the *Binary Scale.* The counter on the unit bar of the multiplier, shows that the whole of the multiplicand is to be set down *once*, as it stands. The next counter, passing over

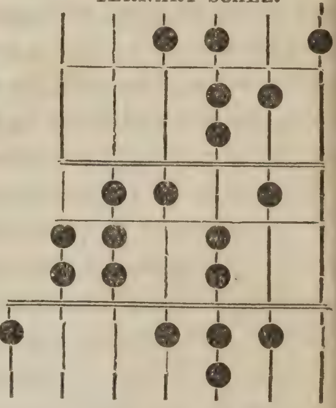
BINARY SCALE.



the vacant bar, indicates by its position, that the whole of the upper range of counters must be advanced *two* bars. The last counter intimates a similar advance to be made again. These various counters are next collected into a single row, which would give by reduction *seven hundred and seventy-seven.*

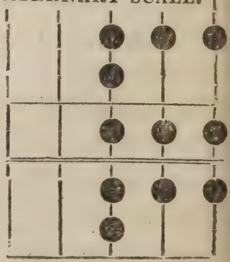
Next, let the same numbers be arranged on the *Ternary Scale.* Here the counter placed on the second bar of the multiplier shows that the counters of the multiplicand are to be carried *one* bar forward. The *two* counters on the third bar, intimate that the range of the multiplicand must be *doubled,* and advanced two bars. The counters on the several bars being now collected and condensed, give this result, composed of *three*, *eighteen*, *twenty-seven*, and *seven hundred and twenty-nine*; making in all *seven hundred and seventy-seven.*

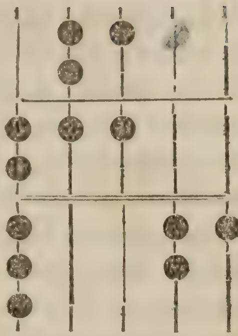
TERNARY SCALE.



On the *Quaternary Scale,* the process will be simpler, and require fewer bars. The *three* successive counters of the multiplier show that the multiplicand is first to be repeated as it now stands, and then at the advance of one and two

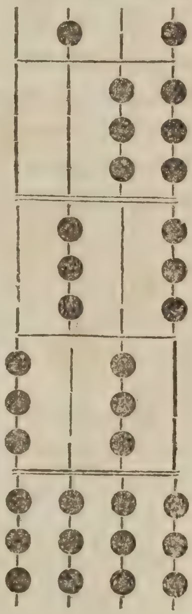
QUATERNARY SCALE.





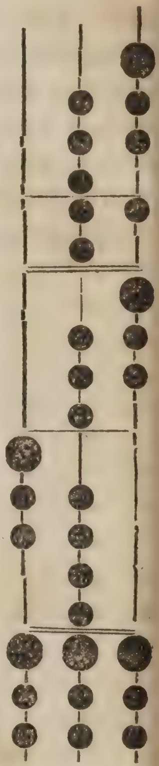
bars. These three successive multiplications give for their collective amount, thrice *two hundred and fifty-six*, twice *four* and *one*, or *seven hundred and seventy-seven*, as in the former examples.

SENARY SCALE.

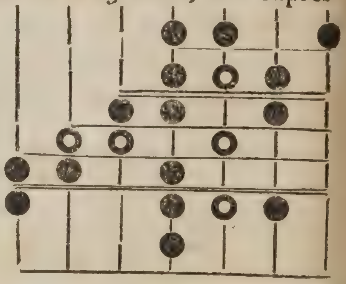


By the *Quinary* and *Senary Scales*, though fewer bars will be required, the operation is on the whole a little more complex. A single instance may be judged sufficient. Thus, the numbers to be multiplied will, on the *Senary Scale*, be represented by *one* counter on the first and the third bars, and by *three* counters on the first and second bars. Consequently the range of the multiplicand must be repeated *thrice* in the order in which it stands, and likewise by *one* bar in advance. The result is, therefore, equal to triple the sum of *one*, *six*, *thirty-six*, and *two hundred and sixteen*, or to *seven hundred and seventy-seven*.

Lastly, let the multiplication of *thirty-seven* by *twenty-one* be performed on the *Denary Scale.* The counter on the unit bar of the multiplier shows that the multiplicand is to be set down *once* in its place, and the two counters of the next bar intimate that it must likewise be *redoubled,* and placed a bar in advance. But the *seven* counters of the multiplicand, on being doubled, leave *four* on the bar of tens, and send *one* to the higher bar, which, with the other *three* counters of the multiplicand likewise doubled, now holds *seven.* These counters being collected, give *seven hundred and seventy-seven.*

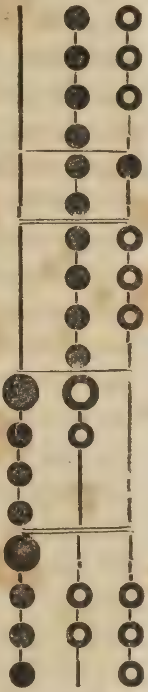


The process of multiplication is often considerably simplified by introducing open or deficient counters. Thus, resuming the example on the *Ternary Scale,* the expression of the multiplier is changed into a single row of counters. The full counter on the second bar shows that the multiplicand is to be advanced by a whole bar, the open counter above it indicates



a similar advance only with inverted counters, and the highest counter intimates that the first operation is merely to be repeated. The several counters are then collected, and again modified into the final result.

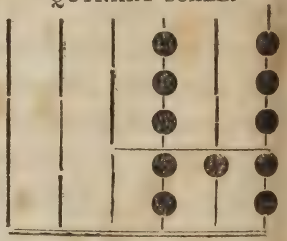
In like manner, the example on the *Denary Scale* is thus modified, the *seven* counters on the unit bar of the multiplicand being exchanged for *three* open counters, with *a full counter* thrown to the next higher bar. This multiplicand is therefore set, as the multiplier indicates, down *once* in the same place, and then *doubled* and shifted a bar higher. *Six* open counters are hence, in the second operation, placed on the second bar, and *eight* full ones moved to the third. Collecting the several counters, the result is *eight hundred*, abating *twenty-three*; that is, *seven hundred and seventy-seven*, as before.

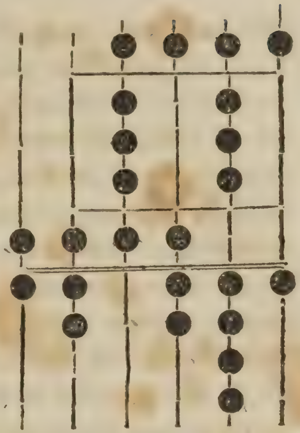


Let another example in Multiplication be taken, where the numbers concerned are rather larger, being *seventy-eight* and *fifty-seven*. Arranged on the *Quaternary Scale*, they will exhibit the form on the following page. Consequently the multiplicand is to be set down *once* in its place, then *doubled* and moved a bar higher, and next *tripled* and advanced another bar. In the second operation, therefore, the *two* right-hand counters of the multipli-

QUATERNARY SCALE.

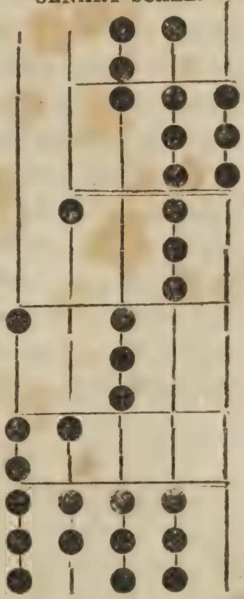
cand would be changed to *four* on the second bar, which is hence left void, and an equivalent counter joined to the doubled *three*, or the *six* of the third bar. On this bar, the *three* of excess are placed, and *one* carried to the empty bars above it. In the third operation, the *two* units of the multiplicand being *tripled*, give an excess of *two*, with *one* thrown to the *nine* of the next bar. On this fourth bar, *two* counters of surplus are laid, and *two* more sent to occupy the fifth bar. All those counters being gathered together on their several bars, exhibit a result which corresponds to *four thousand four hundred and forty-six.*

QUINARY SCALE.

Suppose the same example were transformed to the *Quinary Scale*, it is evident that here the multiplicand must be *doubled*, commencing at the first, and again at the third, bar; and that it must be set



down *once*, beginning at the second bar. The doubling of the *three* counters of the multiplicand on the first bar, gives *a counter* in excess, with *another* to the next bar; and the same process converts the *three* counters of the third bar into *one* counter on that bar, and *another* on the fourth bar. The amount of all these counters afford by reduction the same product as before.

SENARY SCALE.



On the *Senary Scale*, the multiplicand must be successively *tripled*, and then *repeated* two bars in advance. The *triplication* of its *two* highest counters will exactly fill up the corresponding bar; or, what is the same thing, that bar will be left void, and a single representative counter transferred to the next bar higher. The subsequent collecting of the counters requires no condensation, and the whole process in this case becomes extremely simple; the first bar being vacant, the second, third, and fifth being occupied by *three* counters, and the fourth bar holding only *two*.

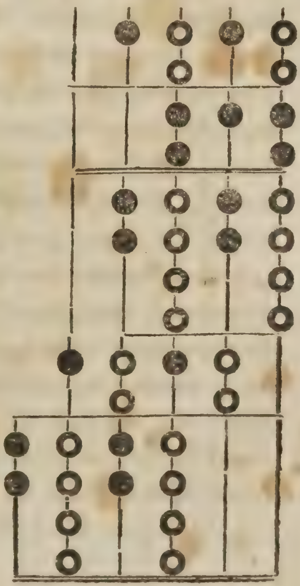
Suppose the same Multiplication were performed on the *Denary Scale.* The *eight* counters on the first bar of the multiplicand being repeated *seven* times, leave *six*, and send *five* counters to the higher bar. The *seven* counters again being repeated as often, furnish, with the addition of those five, an excess of *four*, and throw other *five* to the bar of hundreds. In the second operation, the *eight* counters repeated *five* times, send, without any excess, *four* counters to the third bar; where the *seven* counters being repeated as often, give a farther excess of *five*, and transmit *three* counters to the bar of thousands. Adding together those counters, the *six* counters of the first bar, and the *four* of the second remain unaltered, while the two *fives* on the third bar, leaving the *four* surplus counters, furnish out *another* to the next bar.

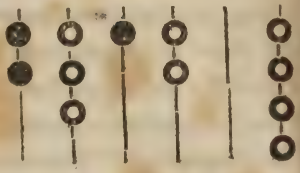
Lastly, let the process be transferred to the *Duodenary Scale.* Here, for the sake of convenience, the multiplier and multiplicand are made to change places. But the *nine* counters on the first bar of the upper number, when repeated *six*

times, give *four* dozen and *six*; and the *four* counters repeated after, make *two* dozen, or leave the *four* advanced counters, and send *two* to the fourth bar. The same operation is again renewed with the other *six* counters, and carried a bar higher. Collecting now the several counters on the bars, the product is, in mercantile language, *two double gross, six gross, ten dozen and six;* which is equivalent, therefore, to the amount of *three thousand, four hundred and fifty-six, eight hundred and sixty-four, an hundred and twenty-six,* making collectively *four thousand four hundred and forty-six.*

Conceive the same operations to be performed by help of deficient counters. On the *Quinary Scale*, the numbers *seventy-eight and fifty-seven*, will stand thus. The former, or the multiplicand, is, therefore, first *doubled*, then repeated a bar higher, and next *doubled* again and advanced two bars. In collecting the counters, *four* open ones appear on the first bar; *two* open and *two* full ones produce a mutual balance, and consequently a vacuity on the second bar; *eight* open and *one* full counter occupy the third bar, leaving a surplus of *two* open counters,

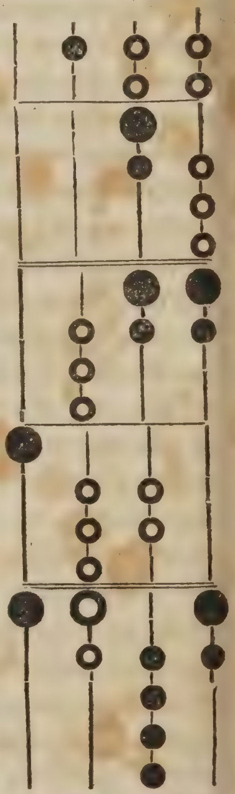
QUINARY SCALE.





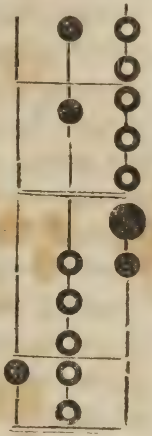
with *another* open counter to be carried to the fourth bar; on which a pair of full counters balance another pair which are open, and the *two* remaining full counters are reduced to a *single one*, by the open counter annexed to them; while, on the fifth bar, the *four* open counters are abridged to *three*, by the influence of the full counter immediately above them. This amount, after reduction, gives the same result as before.

DENARY SCALE.

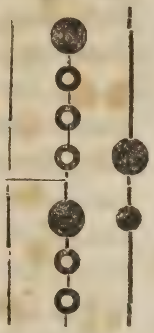


On the *Denary Scale*, the numbers to be multiplied will assume this form. The *three* open counters on the right hand of the multiplier intimate that the counters of the multiplicand are to be *tripled*, and their characters reversed; which gives *six* full counters for the first and second bars, and *three* open counters for the third. Again, the *six* full counters of the multiplier show that all the counters of the multiplicand are to be repeated *six* times, and moved a whole bar in advance. But the *two* open counters repeated so often, give a surplus of *two*, and transmit *one* to augment the product on the next bar, which acquires *three* open counters, and sends another to the fourth bar, where the *six* full counters is reduced by this junction to *five*. The result of the whole is, therefore, *five-thousand and forty-six*, abating *six hundred*.

The application of open or deficient counters will be found useful, in varying and simplifying the process of multiplication, even where smaller numbers are concerned. Thus, to confine our views to the common *Denary Scale*, suppose it were required to multiply *eight* by *seven*. The former, being *two* less than ten, may be denoted by *one* counter on the second bar, and *two open counters* on the first; and the latter, being *three* less than ten, is expressed by *one* counter on the second, and *three open counters* on the first bar. These open counters show that the terms of the upper number should be subtracted *three* times; that is, repeated *thrice* with an opposite character; which gives *six full counters* for the first bar, and *three open ones* for the second. Again, the *full counter* on the second bar indicates that those terms are to be repeated unchanged, but placed one bar in advance.



The combination now made exhibits the product. But, instead of the *single counter* on the third bar, substitute *two* FIVES on the second; the result will then be decomposed into *five*, abating *three*, and *five* abating *two*, reckoned as TENS, together with the counters *on the first bar*, or twice *three*, considered as UNITS.



Hence the explication of the method for multiplying any numbers under ten, by help of the fingers merely; an arithmetical curiosity, probably communicated by the Arabians, but certainly known to the mathematicians of

Europe at the period of the revival of science, though lately introduced again as a novelty, among other improvements, into the practice of elementary schools. In the preceding example, beginning at the left, and thence going to the right hand, *eight* fingers, (including the thumb), are counted, leaving *two* fingers to close. Again, proceeding the reverse way, the number *seven* leaves an excess

of *three* fingers to be shut on the left hand, as in the above disposition of counters. Now, joining the projecting fingers of both hands, or *five* abating *two*, and *five* abating *three*, we obtain *five* TENS, or *fifty*; while the product of the closed fingers or *two* times *three*, gives an accession of *six* UNITS; and consequently the combined result is *fifty-six*.

As another example of this mode, suppose it were desired to multiply *nine* by *six*. These numbers may be severally expressed by *ten* abating *one*, and by *ten* abating *four*. But the *four open counters* of the multiplier signify that the upper counters are to be changed and then repeated *four* times; while the *full counter* intimates that those counters are, without alteration, to be advanced a whole bar. Instead, however, of the single counter on the *third* bar, *two fives* may be combined with the open counters on the *second*. But this procedure can be imitated by the play of the fingers. Counting *nine* fingers, beginning with the left hand, and passing to the right, we leave *one* finger close; and again reckoning *six*, by proceeding from the right to the left, we have *four* fingers remaining to close. If we now bring together the erect fingers of both hands, we shall have, of TENS, *five* abating *four*, and *five* abating *one*, that is, *fifty*; together with the product of the shut fingers, or *four* UNITS told *once*.

This philosophical trick cannot fail to appear striking to young practitioners, and may prove really useful to them, by helping to fix thoroughly and accurately in their memory the ordinary multiplication table. But the same principle might be extended farther. Suppose it were required to multiply *ninety-nine* by *ninety-eight*. These numbers are merely *one hundred*, abating respectively *one* and *two*. The *two open* counters of the multiplier signify that the counters of the multiplicand are to be doubled and reversed, while the *single full* counter intimates that those must also be repeated two bars in advance. The product is, therefore, of *hundreds* one hundred fold, abating *three*,—together with *two* units,—that is, *nine thousand seven hundred and two.*

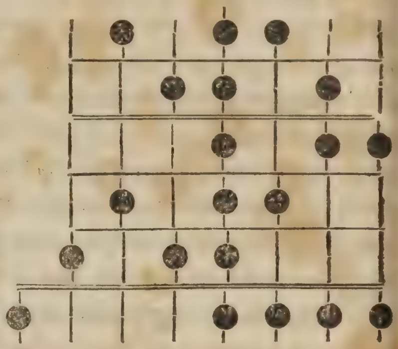
DENARY SCALE.

In this example, the numbers may be conceived as arranged on a new scale, which proceeds by hundreds. This index is hence diminished by *three*, their joint defects, to denote so many hundreds; while *two*, the product of those defects, exhibits the units to be added. But such, we have seen, is the very mode furnished by the fingers on both hands, for multiplying the numbers under ten.

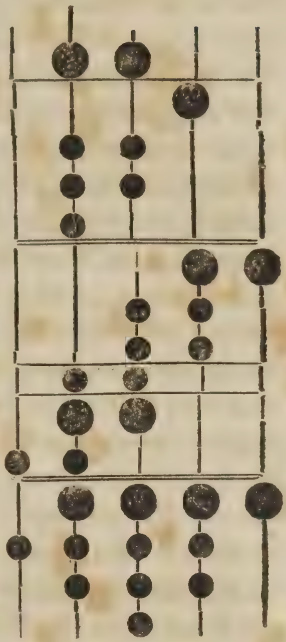
When fractions are expressed on the same numerical scale, their multiplication proceeds with equal facility as that of integers; it being only requisite to commence with the bar of units, and to descend with the lower bars. Thus, if it were sought to multiply *five and a half* by *three and a quarter*. These quantities would evidently, on the *Bi-*

*nary Scale*, be thus expressed. If we start from the bar of units, all the counters of the multiplicand must be repeated in the same position, and the multiplications from the descending bars would only be carried by equal gradations to the right hand. To preserve regularity, therefore, it will be more convenient to begin with the lowest counter, which occupies the *second* descending bar. Hence the whole train of the multiplicand is to be set down *two* bars lower. Then the process goes on as usual; that row of counters is repeated *two* bars in advance, and again at the interval of *three* bars. The counters being now collected together, express *seventeen* and *seven-eights*.

BINARY SCALE.

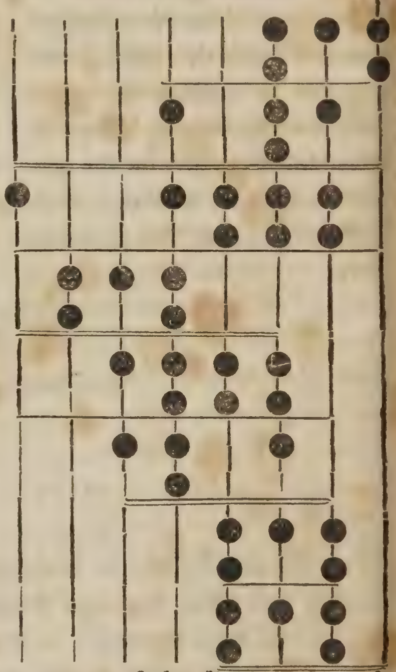


Let the same mixed numbers be represented on the *Quaternary Scale*. Beginning at the right hand, the counter on the first descending bar shows that the whole of the multiplicand must be repeated *one* bar lower; and the *three* counters on the bar of units intimate that it is to be *tripled* in its actual position. The result of the summation of the several bars is the same as before.

Lastly, suppose those quantities were transferred to the *Denary Scale.* The *five* counters of the multiplier on the *second* descending bar, show that the multiplicand is to be repeated as often *two* bars lower; which gives on the descending bars *five* for the third bar, *seven* for the second, and *two* for the first. The next *two* counters double the range, *one* bar in advance; and the *three* highest counters *triple* the whole, at the advance of another bar. All those counters, again, being collected together, give for the product, *seventeen,* with *eight hundred and seventy five-thousand parts.*

DIVISION

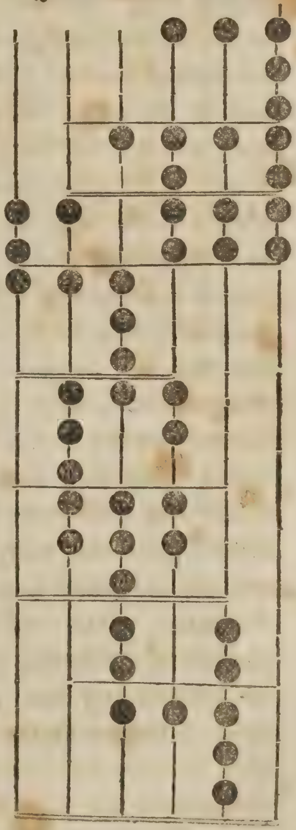
Is the opposite process to Multiplication, and consists in finding how often the same number can be separated or drawn out from another. In the rudest way, therefore, this operation would be performed, by telling over a certain number of counters repeatedly from the same heap. But instead of a slow process of repeated subtraction, the number to be severed, or the *Divisor*, may be first multiplied to approach the mass to be shared, or the *dividend*. The remainder can again be treated in the same manner, and the operation renewed, till nothing is left of the dividend, or a difference less than the divisor itself. Those multipliers, collected together, will express the *quotient*, or the number of subtractions required to exhaust the mass. A few examples shall be selected for illustration. Suppose *two thousand three hundred and forty-six* were to be divided by *twenty-three*. Let these numbers be arranged on the *Ternary Scale*, the *dividend* being the lowermost, and space left for putting the *quotient* immediately under the *divisor*. Beginning at the left hand, it is easy to perceive that the divisor is contained *once* in the first four bars; place *one* counter then for the quotient on the last of these bars; set the divisor directly beneath the dividend, and note the excess, which is *two* counters on the sixth bar, and *two* counters on the fifth. Of this difference, with the next *two* counters of the dividend brought down, three bars are less than the divisor, but four bars

TERNARY SCALE.



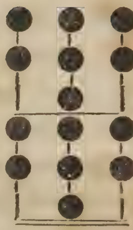
will evidently contain it *twice.* Passing over one bar, therefore, *two* counters joined to the quotient are placed on the third bar of the range. The divisor is then doubled and set down. The remainder of this operation, with the *two* final counters annexed, is exactly the same as the divisor, which must therefore be contained *once;* and here the operation terminates, leaving the first bar vacant. The quotient is hence by reduction equal to *one hundred and two.*

QUATERNARY SCALE.

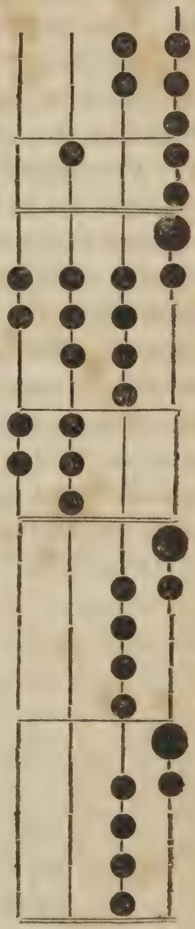


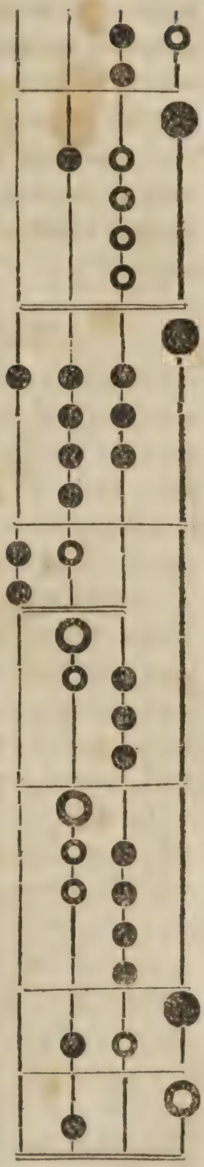
Let the same division be performed on the *Quaternary Scale.* The divisor is evidently contained *once* on the three highest bars of the dividend: *one* counter of the quotient is, therefore, placed on the fourth bar, and the divisor itself set down for subtraction. The remainder is denoted by *three* counters on the fifth bar, *one* on the fourth, and *two* brought from the dividend to the third bar. In this excess, the divisor is contained *twice;* it is consequently doubled and subtracted. But the surplus now, with the two counters annexed from the dividend, contains the divisor *once* on the next bar. The remainder after the third subtraction is the same as the divisor *doubled.*

Whence *two* counters are placed on the last bar of the divisor, and the operation closes. The quotient is, therefore, equal to the amount of *sixty-four*, of *twice sixteen*, of *four*, and of *two*, or *an hundred and two.*



Suppose the same operation were performed on the *Denary Scale.* Here the divisor being identical with the two highest bars of the dividend is, therefore, contained *once*; but it is not contained at all in the *four* counters brought down from the next bar. That bar of the quotient is consequently left vacant; but, from the following bar, the *six* counters are brought down, forming together a number in which the divisor is obviously contained *twice.* But the divisor being *doubled* gives the very same number, and hence leaves no remainder. The collective quotient of the divisor is thus *one hundred* and *two*, as in the preceding examples.



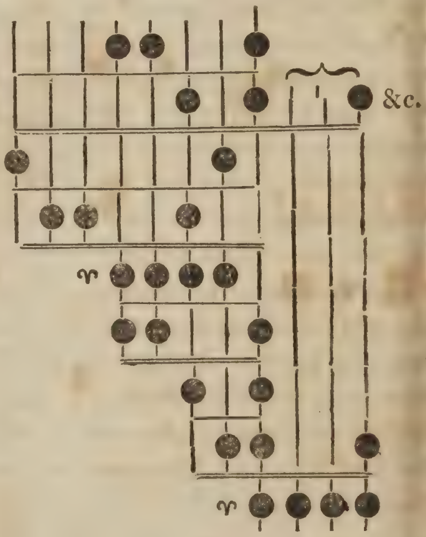


Lastly, let the process be conducted on the *Duodenary Scale.* To avoid prolixity, however, it will answer better in this case to employ the admixture of open counters. Here the divisor may be assumed as *once* contained in the two highest bars of the dividend. Conceive, therefore, a *full counter* to be placed on the fourth bar, and *twelve* correspondent open ones, making an excess of *eight* on the next bar. There consequently remain on that bar *seven* open counters after the first subtraction, to which the *three* full counters are brought down from the dividend on the second bar. In this deficient quantity, the divisor may be considered as contained *four* times, and the quotient is marked by so many open counters. But the counters of the divisor being multiplied by these *four* open ones, give *four* full counters to the second bar, and *eight* open ones to the third. The remainder is hence a full counter on the third bar, and an open one on the second, to which the *six* full counters are subjoined from the first bar of the dividend. In this last surplus, the divisor is contained exactly *six* times, the result of its multiplication being *six* open counters on the first bar, and a full counter on the third, the second bar being passed over; but the open counter on the second bar of the remainder would in effect be taken away, by changing the *six* full counters on the first bar with as many open ones.

This mode of introducing deficient counters is often very convenient in practice, since it only requires, at any step, to know the nearest integral quotient, without regarding whether this be less or greater than the quantity sought.

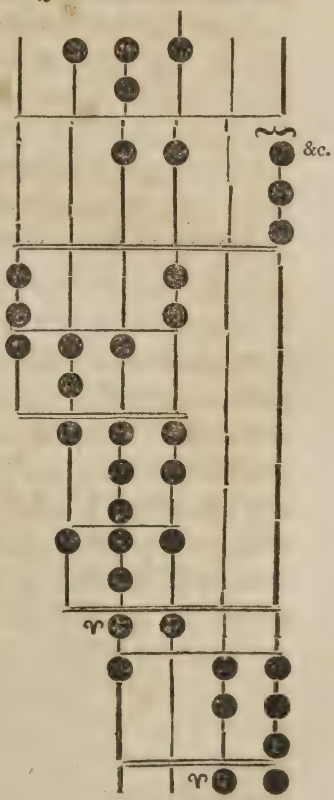
In all the preceding examples, the divisor is complete; but it will often happen that a remainder is left, and consequently that the process may be continued on the descending bars, expressing an excess of a fractional quotient, which either terminates, or constantly recurs again in a perpetual circle. Suppose, for example, it were proposed to divide *one hundred and thirty* by *twenty-five* These numbers would be thus arranged on the

BINARY SCALE.

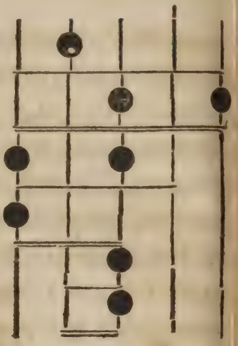



Under the third bar, the divisor is contained *once*, leaving a counter on the seventh, one on the sixth, and another on the second bar. Passing over the second bar, the divisor is contained *once* under the first; but not again till after an interval of *two* bars, when there is left, as under the third bar, *four* consecutive counters. At this point, therefore, a circulation must take place, since the third bar below it corresponds to that of units itself. The same sequence will be continually maintained: First an empty bar, then a counter, followed by two empty bars.—To indicate this circle of renovation, the mark ♈ for *Aries*, the first Sign of the Ecliptic, is adopted, as intimating the birth of the revolving year; and, therefore, by extension, the recommencement of a periodical cycle.

QUATERNARY SCALE.




Let the same process be transferred to the *Quaternary Scale*. The divisor is contained *once* in the three highest bars of the dividend; and the subtraction being made, there remains *one* counter on the third bar, and *three* counters on the second, to which are subjoined on the first bar the *two* counters brought down from the dividend. In this quantity, the divisor is contained once again; and *one* counter is left on the second, and *another* on the first bar. Now, passing over two bars, the divisor is contained *three* times, with the same remainder of two consecutive counters. Wherefore the operation is continually renewed, and *three* counters run through the whole train of subsequent bars.



Lastly, suppose this division were performed on the *Quinary Scale.* The divisor occurs *once* in the first bar of the dividend, and *once* again, after an interval of two bars, in the remainder. In short, the quotient here is precisely the same as the dividend, only placed two bars lower. The result is consequently, as before, *five* and *one-fifth.*

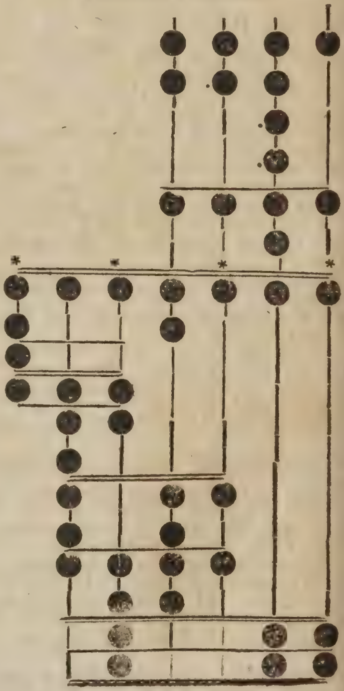
In these last examples, the integral part of the quotient may be considered as forming a distinct number; while the remainder of the division constitutes the numerator of a fraction, of which the divisor is the denominator, the conversion or development of it along the range of inferior bars being effected in the way formerly explained in treating of Numeration.

We have thus explained, at some length, the modes of performing the four common Rules of Arithmetic, by means of counters. But the most complex calculations in which numbers are concerned being all reducible to such elementary operations, it seems unnecessary to descend to the details generally given, under the various heads of *Proportion, Fellowship, Interest,* or *Exchange.* In the application of Arithmetic, however, to Practical Geometry, the process of Extracting Roots becomes farther indispensable.—This will require some explanation. As the repeated multiplication of the same number by itself, forms its successive Powers; so, in reference to these again, the number which generates them is called their Root. To find any power of a number is hence an easy operation; but the converse of this problem, or the extracting of a root, that is, the discovering of the number from whose involution or repeated multiplication the given power had arisen, can be effected only by a sort of tentative procedure, which is in some cases attended with considerable difficulty. We shall, therefore, confine our views at present to finding the Square Root, or the evolving of the Root of the Second Power.

In order to investigate the method of proceeding, we have only to consider how the second power of a compound number is formed. Conceive this number to consist of two members, or to be represented on two consecutive bars of any scale. The same number being repeated as a multiplier, it is evident that their product, that is, the square or second power, must, following the order of the multiplication, consist of the square of the first member, the product of it into the second, and again this product by the second, together with the square of this member. Consequently, the square of the compound number is analysed into the square of its first member, and the product of twice that member joined to the second, by the same second member. Having substracted from the given number, therefore, the square of the first member of the root, the remainder is to be divided by twice that member augmented by the quotient itself, to find the second member. But if the division should not terminate, it is evident that the process of exhaustion may be still continued; for, considering the numbers exhibited on the two bars as condensed into one group, the addition or correction on the following bar is discovered by the same mode of decomposing the residual portion. It is farther evident, from the practice of multiplication in the forming of powers, that each bar of the root must correspond to a pair of bars on the square.

The whole procedure in the extraction of the square root will be readily understood from a few select examples. Suppose it were sought to discover the square root of *eighteen hundred and forty-nine.* This number expressed on the *Ternary Scale* will stand as below, occupying partially *four* pair of bars, noted by so many asterisks. Hence the root must likewise be contained on *four* bars. Beginning, therefore, with the *two* counters of the highest pair, the nearest root is obviously *one.* A counter being hence placed on the fourth bar, its square is subtracted from those *two*, and the remainder conjoined with the counters brought down from the next pair of bars give a single row of *three* counters. Let this be now divided by the *double* of the first member of the root, to discover the second or additional member; but before the division is completed, annex the quotient itself to the divisor. Wherefore, above the counter of the root, and on the same bar, place *two* counters, which, being contained *once* in the row of *three* counters to be decomposed, set on the next bar *one* counter for the root, and another for the divisor. This compound is then multiplied by the annexed counter of the root, and is consequently set down two bars higher, or beginning with the fifth

TERNARY SCALE.

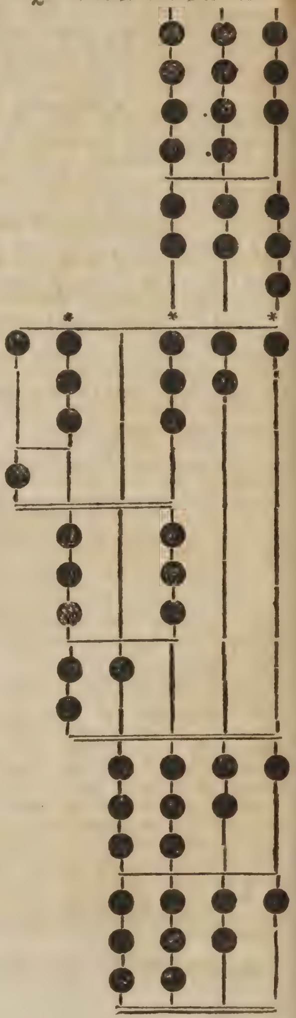


bar. The subtraction being now performed, there are left *two* counters on the sixth, to which is subjoined the next period, consisting of *two* counters on the third, and *one* on the second bar. For a third operation, then, the two highest bars of the root, viewed as consolidated into a single member, are *doubled*, which is done by repeating the counter on the third bar of the divisor, a point on the left side of it indicating this accession, as a point on the other side of a counter was employed to signify its being withdrawn. This new divisor is contained twice in the remainder; and consequently *two* counters are placed on the second bar, after the portion of the root already found, and after the divisor itself. The divisor being now multiplied by those *two* counters of the root, and subtracted, leaves a *single* counter on the fifth bar, to which is annexed the *two* single counters of the last period. To decompose this remainder, the counters on the three bars of the root, considered as forming one cluster, are *doubled* for a final divisor, which is effected by subscribing *two* doted counters below the other *two* on the second bar. The divisor now occurs only once, and therefore *a counter* is annexed to it on the first bar, and likewise to the root. But there is at last no remainder; for the *four* counters on the second bar of the divisor leave *one* and send one to the two counters of the next bar; which being then full, transmit a counter to the fourth bar, whence another counter is sent up to the fifth. This complex process of extraction is therefore completed, and the root sought is the *Ternary Expression* for the number *forty-three.*

Let the number to be extracted be arranged on the *Quaternary Scale.* It will occupy *three* pairs of bars, and consequently its root must likewise stand on *three* bars.

Of the first period, consisting of *a counter* on the sixth bar, and *three* counters on the fifth, the nearest inferior square root is evidently *two*; which being squared, gives *four* counters for the fifth bar, or *one* counter for the sixth. There are left after the subtraction, *three* counters on the fifth bar, to which are subjoined, on the third bar, *three* counters brought down. In this remainder, the *four* counters of the third bar, or the first part of the root *doubled*, is contained *twice*. On the second bar, therefore, *two* counters are placed, both after the root and after the divisor, which is multiplied by them. The product, expressed by *two* counters on the fifth bar, and *one* on the fourth, is next subtracted, leaving the fourth and third bars each occupied by *three* counters. The last period, consisting of *two* counters on the second, and *one* on the first bar, is now annexed, to complete the dividend. The two first bars of the root, considered as a distinct group, are likewise *doubled* for the divisor, by repeating with dots prefixed the *two* counters of the second bar.

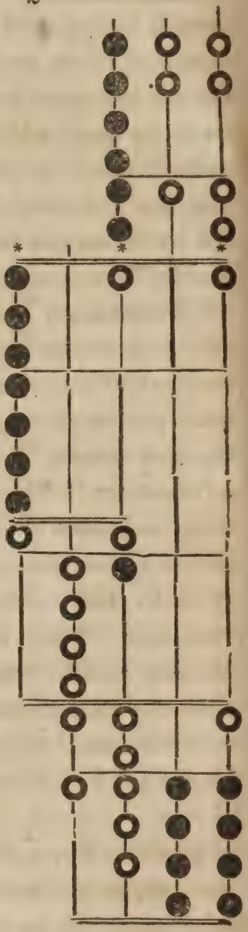
QUATERNARY SCALE.



This compound divisor is contained *thrice*; and, consequently, having *three* counters annexed to it, the whole is multiplied by the same *three* subjoined likewise to the root. The product gives *one* counter to the first bar, and sends *two* to the next, which hence acquires *two* counters, and transmits *three* to the third bar, where these *three* are dropped, and *three* more conveyed to the fourth bar. The operation consequently terminates, and the root thus obtained corresponds, as before, to *forty-three*.

Not to multiply illustrations, let the same process be performed on the *Quinary Scale*, employing, however, *open counters* for the sake of simplicity. The original number of which the root is to be extracted here occupies partially *three* pair of bars. But the *three* counters of the highest period have *two* for their nearest superior root. Consequently, while *two* counters are placed on the third bar as the first portion of the root, *four* counters, being their square, are set under those *three*. The subtraction leaves *an open* counter on the fifth bar, to which is now subjoined *another open* counter brought down on the third bar. In this remainder the divisor, or *doubled* member of the root, is contained *once*; and consequently *an open* counter is on the second bar annexed to that divisor, and likewise to the root. But this *open counter* being multiplied into the compound divisor, first changes the *open counter* above it into a *full one*, and then converts the *four open* counters preceding it into as many *full counters*. The subtraction therefore leaves *an open* counter on the fourth bar, and *two* such on the third, to which are subjoined a *single open* counter on the first bar. The two highest bars of the root, viewed as forming a distinct member of it, are then *dou-*

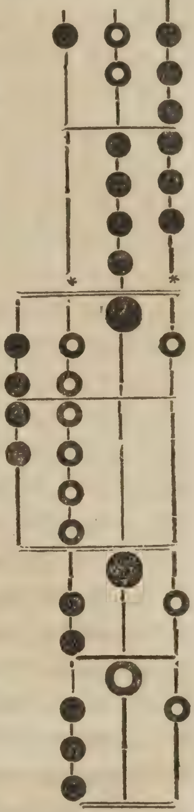
QUINARY SCALE.



*bled*, for the divisor; that is, the *open* counter on the second bar is again repeated with a dot placed before it. But this compound divisor being now contained *twice* in the remainder, *two open* counters on the first bar are annexed to the root, and again placed above it. The product of the multiplication by those *two* lower *open* counters consists, therefore, of *four* full counters on the first and second bars, and *eight open* counters on the third bar, which leave *three* on that bar, and deliver *another open* counter to the fourth. There is, however, no remainder; for *an open* counter joined to the *two* on the third bar of the minuend would be balanced by *five full ones* on the second bar, leaving, consequently, another *full one*, which is equivalent to *five* such on the first bar where the open counter reduces them to *four*, the very same as in the subtrahend.

Lastly, in performing this extraction on the *Denary Scale*, the notation at least will be somewhat abbreviated, by adopting *open counters*. The given number now ranges on *two* pair of bars; but of the highest period, the nearest root is *four*, the square of which, or *sixteen*, denoted by *two full* counters on the fourth bar, and *four open* ones on the third, being subtracted, leave *two open* counters. The next period consisting of *five full* counters on the second bar, and *an open one* on the first, is then subjoined to the *two full* counters on the third. To obtain the corresponding divisor, the *four* counters of the root are *doubled*, and consequently are expressed, by placing *two open* counters on the same bar, and a *full one* on the third bar. The divisor being contained *thrice*, in this remainder, *three* counters are accordingly placed both after that divisor and after the root itself. The product of the multiplicand, though differently expressed, is obviously the same as the minuend. The operation ends here, and the square root sought is consequently *forty-three*.

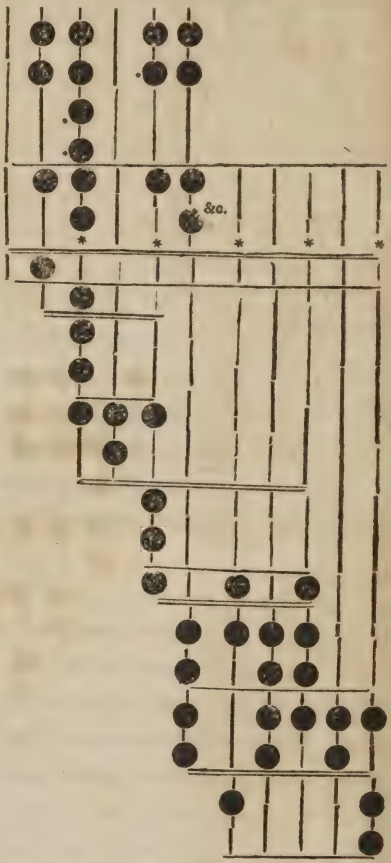
DENARY SCALE.



It is easy to perceive, that if the process of decomposition should not terminate, the extraction of the square root may, on the same principles, be pursued through all the descending bars, including a pair of those bars at each step. But to elucidate more clearly the mode of procedure when fractions are concerned, we shall take one or two examples, which involve no integral part. Thus, suppose it were required to discover an approximation to the square root of the fraction *one-third*. Represented on the *Ternary*

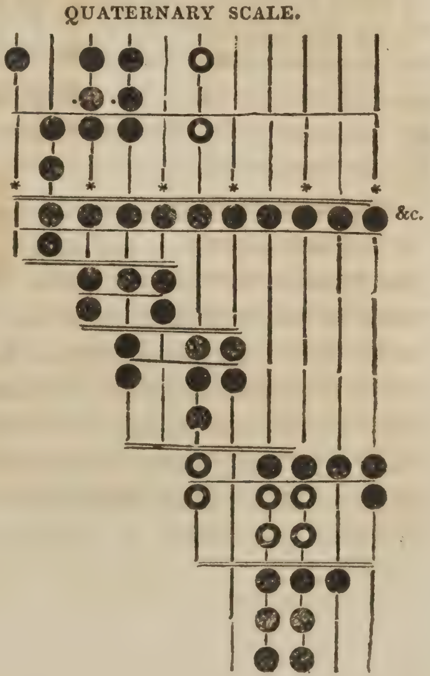
*Scale*, this would evidently be denoted by a single counter on the first *descending* bar. Let every second bar, beginning with that of units, be marked by asterisks, to distinguish the successive pairs in descent. The counter on the second bar, corresponding to *three* on the next bar, has consequently *one* for its nearest root. The square of this, again, throws *one* to the third bar, where, consequently, *two* counters are left after subtraction. The counter of the root is now *doubled* for the divisor; and, being contained *twice* in the remainder, *two* counters are subjoined, both to the root and to the divisor itself. Multiplying now that compound divisor by the *two* counters below it, *four* counters are thrown to the fourth descending bar, which drop *one*, therefore, and send *another* to the bar above, where the *five* counters leave *two*, and convey *one* to the second bar. On subtracting this product, only *two* counters remain on the fourth bar; but the root, after having been *doubled* for a new divisor, is not

TERNARY SCALE.



contained in excess, till after an interval of two periods, when it occurs *once*. A *single* counter, therefore, on the fourth bar, is subjoined both to the root and to the divisor, which is all repeated four bars lower, or condensed into the expression of single counters on the fourth, sixth and eighth bars. The remainder of this third subtraction is considerable, being signified by *two* counters on the fifth bar, *one* on the sixth and *two* on the seventh and on the eighth bar. In this quantity, the corresponding divisor is contained *twice*; wherefore *two* counters on the fifth bar are annexed both to that divisor and to the root, and, the multiplication being performed, leaves yet a small remainder. The process of decomposition may, therefore, be continued indefinitely, though it has already approximated within the five hundredth part of the truth.

Another example may be deemed sufficient. Let the fraction *one-third* be transferred to the *Quaternary Scale*, and it will evidently be denoted by a train of *single counters*, beginning at the second, and running down thro' all the rest of the bars. Consequently the nearest root of the first period is *two*, which, being squared, give *four* to the third bar, or *one* to the second. The dif-



ference, with the next period subjoined, exhibits *three* successive counters. The root being now doubled for a divisor, gives *four to* stand on the second bar, or *one* on the first; which, being obviously contained *once* on the remainder, *a counter* is, on the third bar, annexed to the root, and likewise to the divisor itself. This divisor is, therefore, set down two bars lower, and, being subtracted, leaves a counter on the fourth bar, to which the next period is subjoined. Here the modified divisor is very nearly contained once again, and leaves an *open* counter on the sixth bar, which admits of no division till the two succeeding periods are joined to it. In this compound quantity the divisor is once found, and consequently *an open* counter on the sixth bar is annexed both to the root and to the divisor itself. There is still a small remainder; but the extraction has already been pushed so far as to approximate within the five thousandth part of the true root.

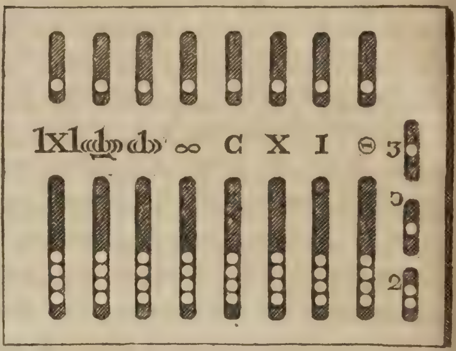
Such is the natural process of analysing numbers, and of variously combining and separating them; and such are likewise the simpler modes of abridging the labour of computation. From the copious illustrations which have been given, it appears that Palpable Arithmetic is capable, if skilfully conducted, of being applied, with considerable facility, to a wide range of combinations. All nations have, accordingly, at different periods, employed that symbolical method of Calculation, which is indeed perpetuated in the term itself. The Egyptians performed their computations merely by the help of pebbles; and so did the Greeks for a lapse of ages. But while the latter, as Herodotus acquaints us, proceeded in such operations from left to right, always descending from the higher part of the number expressed, the former used, both in writing and counting, to advance in the opposite direction, or from right to left, as still practised very generally over the East. In the schools of ancient Greece, the boys acquired the elements of knowledge by working on the *Abax*, a smooth board with a narrow rim; so named, evidently, from the combination of the three first letters of their Alphabet, and resembling the tablet, likewise called A, B, C, on which the children with us were accustomed to begin to learn the art of reading. The pupils, in those remote times, were instructed to calculate, by forming progressive rows of counters, which, according to the wealth or fancy of the individual, consisted of small pebbles, of round bits of bone or ivory, or even of silver coins. From the Greek and Latin word for a *pebble*, comes, in either language, the verb signifying to *compute*. The same board, strewed with fine green sand, a colour soft and agreeable to the eye, served equally for teaching the rudiments of writing and the principles of Geometry.

To their calculating board, the ancients make frequent allusions. It appears, that the practice of bestowing on pebbles an artificial value, according to the rank or place which they occupied, remounts higher than the age of Solon, the great reformer and legislator of the Athenian commonwealth. This sagacious observer and disinterested statesman, who was however no admirer of regal government, used to compare the passive ministers of Kings to the counters or pebbles of Arithmeticians, which, according to the place they hold, are sometimes most important, and at other times utterly insignificant. The Grecian orators, in speaking of balanced accounts, picture this termination, by saying that *the pebbles were cleared away, and none left.* It is evident, therefore, that the ancients, in keeping their accounts, did not separately arrange the credits and the debts, but set down pebbles for the former, and took up others for the latter. As soon as the board became cleared, the opposite claims were exactly balanced.—It may be observed, that the common phrase *to clear one's scores or accounts,* meaning to settle or adjust them, still preserved in the popular language of Europe, was certainly suggested by the same practice of reckoning with counters, which prevailed indeed until a comparatively late period.

The Romans borrowed their *Abacus* from the Greeks, and never aspired higher in the pursuit of numerical science. To each pebble or counter required for that board, they gave the name of *calculus*, a diminutive formed from the word signifying *a white stone;* and applied the verb *calculare*, to express the operation of combining or separating such pebbles or counters. Hence innumerable allusions by the Latin authors. The use of the *Abacus*, called sometimes likewise the *Mensa Pythagorica*, formed an essential part of the education of every noble youth. A small box or coffer, called a *Loculus*, having compartments for holding the *calculi* or counters, was considered as a necessary appendage. Instead of carrying a slate and satchel, as in modern times, the Roman boy was accustomed to trudge to school, loaded with those ruder implements—his arithmetical board, and his box of counters.

In the progress of luxury, *tali*, or dies made of ivory, were used instead of pebbles, and small silver coins came to supply the place of counters. Under the Emperors, every patrician living in a spacious mansion, and indulging in all the pomp and splendour of eastern princes, generally entertained, for various functions, a numerous train of foreign slaves or freedmen in his palace. Of these, the *librarius* or *miniculator*, was employed in teaching the children their letters; but the *notarius* registered expenses, the *rationarius* adjusted and settled accounts, and the *tabularius* or *calculator*, working with his counters and board, performed what computations might be required.

To facilitate the working by counters, the construction of the *Abacus* was afterwards improved. Instead of the perpendicular lines or bars, the board had its surface divided by sets of parallel grooves, by stretched wires, or even by successive rows of holes. It was easy to move small counters in the grooves, to slide perforated heads along the wires, or to stick large knobs or round-headed nails in the different holes. To diminish the number of marks required, every column was surmounted by a shorter one, wherein each counter had the same value as five of the ordinary kind, being half the index of the Denary Scale. The *Abacus*, instead of wood, was often, for the sake of convenience and durability, made of metal, frequently brass, and sometimes silver. Two varieties of this instrument seem to have been used by the Romans. Both of them are delineated from antique monuments,—the first kind by Ursinus, and the second by Marcus Velserus.



In the former, the numbers are represented by flattish perforated beads, ranged on parallel wires; and, in the latter, they are signified by small round counters moving in parallel grooves, as represented in the figure annexed. These instruments contain each seven capital divisions, expressing in regular order *units*, *tens*, *hundreds*, *thousands*, *ten thousands*, *hundred thousands*, and *millions*. For the sake of abbreviation, a similar set of shorter grooves, following the same progression, but having *five* times the relative value, are made to range immediately above them. With *four* beads on each of the long grooves or wires, and a *single* bead on every corresponding short one, it is evident that any number could be expressed, as far as *ten millions*.

In the Roman *Abacus*, the arrangement of the *Denary Scale* is uniformly followed; but there is, besides, a small appendage to the subdivisions, founded on the *Duodenary System*. Immediately below the place of units, is added a bar, with its corresponding branch, both marked ⊖, being designed to signify *ounces*, or the twelfth parts of a pound. *Five* beads on the long wire, and *one* bead on the short wire, equivalent now to *six*, would therefore denote *eleven* ounces. To express the simpler fractions of an ounce, three very short bars are annexed behind the rest; a bead on the one marked S or Ʒ, the contraction for *Semissis*, denoting *half-an-ounce;* a bead on the other, which is marked by the inverted Ɔ, the contraction for *Sicilicum*, signifying the *quarter of an ounce;* and a bead on the last very short bar, marked Ϙ, a contraction for the symbol ϙ or *Binæ Sextulæ*, intimating a *duella* or *two-sixths*, that is, *the third part of an ounce.*—The second form of the *abacus*, here delineated, differs in no esssential respect from the first, grooves merely supplying the place of parallel wires, and coins, or ivory counters, being substituted for the perforated beads.

The Romans likewise applied the same word *Abacus*, to signify an article of luxurious furniture, resembling in shape the arithmetical board, but often highly ornamental, and destined for a very different purpose,—the relaxation and the amusement of the opulent. It was used in a game apparently similar to that of chess, which displayed a lively image of the struggles and vicissitudes of war. The infamous and abandoned Nero took particular delight in this sort of play, and drove along the surface of the *Abacus* with a beautiful *quadriga*, or chariot of ivory.

The Chinese have, from the remotest ages, used in all their computations an instrument similar in shape and construction to the *Abacus* of the Romans, but more complete and uniform. It is admirably fitted for representing the decimal system of measures, weights, and coins, which prevails throughout their vast Empire. The calculator could begin at any particular bar, and reckon, with the same facility either upwards or downwards, through the whole range, which includes ten bars. This advantage of treating fractions exactly like integers, is of the utmost consequence in practice. Accordingly, those arithmetical machines, of very different sizes, have been adopted by all ranks, from the man of letters to the humblest shopkeeper, and are constantly used in all the bazars and booths of Canton and other cities, being handled, it is said, by the native traders with a rapidity and address which quite astonish the European factors.

The civil arts of Rome were communicated to other nations by the tide of victory, and maintained through the vigour and firmness of her imperial sway. But the simpler and more useful improvements survived the wreck of empire, among the various people again restored by fortune to their barbarous independence. In all transactions wherein money was concerned, it was found convenient to follow the procedure of the *Abacus*, in representing numbers by counters placed in parallel rows. During the middle ages, it became the usual practice over Europe for merchants, auditors of accounts, or judges appointed to decide in matters of revenue, to appear on a covered *bank* or *bench*, so called from an old Saxon or Franconian word signifying a *seat*. The term *scaccarium*, a Latinised oriental word, from which was derived the French, and thence the English, name for the *Exchequer*, anciently indicated merely a *chess-board*, being formed from *scaccum*, denoting one of the moveable pieces in that intricate game. The reason of this application of the term is sufficiently obvious.

The Court of Exchequer, which takes cognisance of all questions of revenue, was introduced into England by the Norman Conquest. Fitz-Nigel, in a dialogue on the subject, written about the middle of the twelfth century, says that the *scaccarium* was a quadrangular table about ten feet long and five feet broad, with a ledge or border about four inches high, to prevent any thing from rolling over, and was surrounded on all sides by seats for the judges, the tellers, and other officers. It was covered every year, after the term of Easter, with fresh black cloth, divided by perpendicular white lines, or distinctures, at intervals of about a foot or a palm, and again parted by similar transverse lines. In reckoning accounts, they proceeded, he subjoins, according to the rules of arithmetic, using small coins for counters. The lowest bar exhibited *pence*, the one above it *shillings*, the next *pounds;* and the higher bars denoted successively *tens*, *twenties*, *hundreds*, *thousands*, and *ten thousands* of pounds; though, in those early times of penury and severe economy, it very seldom happened that so large a sum as the last ever came to be reckoned. The first bar, therefore, advanced by *dozens*, the second and third by *scores*, and the rest of the stock of bars by the multiples of *ten*. The teller sat about the middle of the table; on his right hand, *eleven* pennies were heaped on the first bar, and a pile of *nineteen* shillings on the second; while a quantity of pounds was collected opposite to him, on the third bar. For the sake of expedition, he might employ a different mark, to represent half the value of any bar, a silver penny for ten shillings, and a gold penny for ten pounds.

In early times, a *chequered board*, the emblem of calculation, was hung out, to indicate an office for changing money. It was afterwards adopted as the sign of an inn or *hostelry*, where victuals were sold, or strangers lodged and entertained. We may perceive, particularly in the South, traces of that ancient practice existing even at present.

The use of the smaller *abacus* in assisting numerical computation was not unknown during the middle ages. In England, however, it appears to have scarcely entered into actual practice, being mostly confined to those few individuals, who, in such a benighted period, passed for men of science and learning. The calculator was styled, in correct Latinity, *abacista*; but, in the Italian dialect, *abbachista*, or *abbachiere*. A different appellation was afterwards introduced by the Arabians, who conquered Spain, and enriched that insulated country by commendable industry, where they likewise introduced their mathematical science. Having adopted an improved species of numeration, to which they gave the barbarous name of *algarismus*, *algorismus*, or *algorithmus*, from their definite article *al*, and the Greek word for *number*, this compound term was adopted by the Christians of the West, in their admiration of superior skill, to signify calculation in general, long before the peculiar method of performing it had become known and practised among them. The term *algarism* was corrupted in English into *augrim* or *awgrym*, and applied even to the pebbles or counters used in ordinary calculation. The same word, *algorithm*, is now applied by mathematicians, to express any peculiar sort of notation.

The *Abacus*, with its symbolical furniture, had been adopted merely as an instrument for expediting the process of computation. But it became likewise necessary to have recourse to some readier and simpler modes of expressing numbers. A very ancient practice, though quite arbitrary in its principle, consisted in employing the various disposition of the fingers and the hands, to signify the numerical series. On this narrow basis, a system was framed of very considerable extent.

By a single inflexion of the fingers of the left hand, the Romans proceeded as far as *ten*; and by combining another inflexion with it, they could advance to *an hundred*. On the right hand, the same signs being augmented an hundred fold, carried them as far as *a thousand*, and *ten thousand;* and, by another extension, those signs variously referred to the head, the throat;—or on either side, to the breast, the stomach, the waist, or the thigh, were again multiplied an hundred times, and consequently raised to the extreme limit of a *million*.—Thus, with the fingers of the left hand, the first set of inflexions, as in the specimen here annexed, denoted the nine *digits*, and the second set of them represented the nine *decads;* but, performed on the right hand,


the former disposition of the fingers indicated *hundreds*, and the latter was made to signify *thousands*.

In this numerical play, the Romans especially had, from constant experience, acquired great dexterity and address. Many allusions to the practice are made in the writings of their poets and orators; and, without some knowledge of the principle adopted, many passages of the classics would lose their whole force. This kind of pantomime even outlived the subversion of the Western Empire, and was particularly suited to the slothful habits of the religious orders who fattened on its ruins, and, relinquishing every manly pursuit, recommended silence as a virtue, or enjoined it as a duty.

These signs were merely fugitive, and it became necessary to adopt other marks, of a permanent nature, for the purpose of recording numbers. But of all the contrivances adopted with this view, the rudest undoubtedly is the method of registering by *tallies*, introduced into England along with the Court of Exchequer, as another badge of the Norman Conquest. These consist of straight well-seasoned sticks, of hazel or willow, so called from the French verb *tailler*, *to cut*, because they are squared at each end. The sum of money was marked on the side with notches, by the cutter of tallies, and likewise inscribed on both sides in Roman characters, by the writer of the tallies. The smallest notch signified a penny, a larger one a shilling, and one still larger a pound; but other notches, increasing successively in breadth, were made to denote *ten*, *a hundred*, and *a thousand*. The stick was then cleft through the middle by the deputy-chamberlains, with a knife and a mallet; the one portion being called the *tally*, or sometimes the *scachia*, *stipes*, or *kancia*; and the other portion named the *counter-tally*, or *folium*.

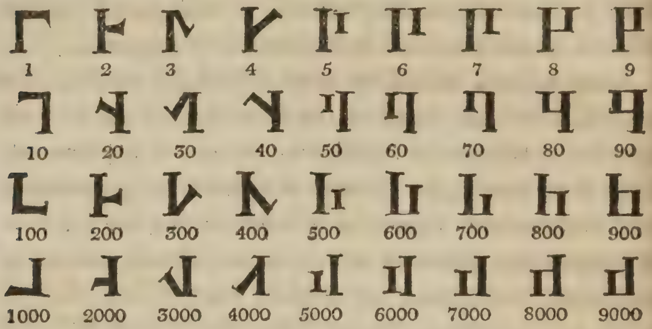
This strange custom might seem the practice of untutored Indians, and can be compared only to the rude simplicity of the ancient Romans, who kept their diary by means of *lapilli* or small pebbles, casting a white pebble into the urn on fortunate days, and dropping a black one when matters looked unprosperous; and who sent, at the close of each revolving year, their Praetor Maximus with great solemnity to drive a nail in the door of the right side of the temple of Jupiter, and next to that of Minerva, the patron of learning, and the inventor of numbers.

# FIGURATE ARITHMETIC.

THE science of number received its capital improvement in the adoption of a ready and comprehensive system of characters, not only qualified to exhibit directly the widest range of objects, but also fitted as instruments for facilitating every process of calculation. This might appear an easy step of advancement from the practice of the *Abacus*, since it was only requisite to devise the few symbols wanted in expressing the counters that could occupy any single bar of the common or denary scale. But the simplest and most valuable discoveries are seldom the first achieved. The transition made in the art of writing from the use of emblematic figures, to that of arbitrary signs chosen to express merely the sounds of a conventional language, must have, on the whole, contributed to retard the progress of numeral notation. The system of characters among the Romans, especially after it had changed its primitive forms into the letters most analogous, was so complex and unmanageable, as to reduce them to the necessity, in all cases, of employing the *Abacus*. The structure of the Greek numerals, though founded entirely on the distribution of the alphabet, was more pliant and refined. Three sets of letters carried the direct notation to a *thousand;* and, by punctuating the first series, its power was extended as far as *ten thousand*, or four places of the denary scale. To represent the largest numbers, the principle of position, after having been so long abandoned, was again resumed, and the same quaternion or period of progressive *myriads* was repeated in continual ascent. In this improved arrangement, the value of the characters depending on their place or rank, it became essential to mark or fill up the accidental vacuities in the scale of numeration, and a small *o*, insignificant by itself, was accordingly adopted for that very purpose.

The system of numerals, thus finally moulded by the Greek astronomers, though cumbrous and redundant in its structure, had therefore attained a high degree of perfection, and was capable, with due labour and patience, of performing the most complex operations in Arithmetic. The extent of their alphabet was favourable to the first attempts at numeration; since, with the help of only three intercalations, it furnished characters for the whole range below a thousand. This very circumstance, however, proved a bar to future improvements. The first series of letters was already distinguished from the two succeeding classes, in being employed with subscribed points to denote thousands, and complete the quaternary period. It might hence appear as no violent, yet a most important, innovation, if only those letters denoting the nine digits had been retained, and the rest signifying tens and hundreds entirely dismissed. By such a change, the arithmetical notation of the Greeks would have reached its utmost term of simplification, and have exactly resembled indeed our own. Had the genius of that people not suffered a fatal eclipse, they must have soon passed the few barriers which remained to obstruct their progress.

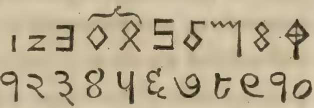
Some writers, misled by very superficial views of the subject, have still ascribed the invention of the modern numeral characters to the Greeks, or even to the Romans. Both these people, for the sake of expedition, occasionally used contractions, especially in representing the fractions of weights and measures, which, to a credulous peruser of mutilated inscriptions, or ancient blurred manuscripts, might appear to resemble the forms of our ciphers. But this resemblance is merely casual, and very far indeed from indicating the adoption of a regular denary notation. The most contracted sort of the Roman writing was formed by the marks attributed to the freedman Tiro, and to Seneca the philosopher, while that of the Greeks was mixed with the symbols called *Siglæ;* both of which have exercised the patience and skill of our Antiquaries and Diplomatists. In the latter species of characters, were kept the accounts of the revenues of the Empress Irene at Constantinople. The Arabians and Persians likewise employed a concise but arbitrary mode of representing the larger numbers, by means of abbreviated words, a practice which still prevails in the East. The Chaldean astronomers, and their successors, in the Lower Empire, although acquainted with the simple and elegant system of the ancient Greek notation, yet preferred in certain cases a sort of tachygraphical marks, extremely abbreviated, but entirely conventional and arbitrary in their formation. These artificial characters were formed merely by a single broad stroke, with a smaller branch variously inserted, as in the specimen below:



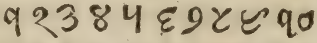

*One million* was denoted, by merely doubling the character for a thousand.—The principal stroke could have either a perpendicular or an horizontal position; which gave occasion to two distinct ways of tracing those marks.

Such, undoubtedly, were the characters which John Basingstoke, archdeacon of Leicester, according to the relation of Matthew Paris or his continuator, brought into England, and communicated to some of his friends, as a precious acquisition, shortly before his death in 1252. Had that eminent person, whom the thirst of knowledge in a dark age led to visit the East, and to study the Greek language amidst the ruins of Athens, been at all acquainted with the nature of the Arabic numerals, he must have perceived the comparative futility of every plan which aimed at mere abbreviation.

It cannot be doubted, that we derived our knowledge of the numeral digits from the Arabians, who call the system *Hindasi*, and confess their having obtained this invaluable acquisition from an extended intercourse with the East. But for want of precise historical evidence, much obscurity still hangs over the whole subject. If the exuberant fancy of the Greeks led them far beyond the denary notation, it seems probable, that the feebler genius of the Hindus might just reach that desirable point, without diverging into an excursive flight. Though now familiar with that system, they are still unacquainted with the use of its descending decimal scale; and their management of fractions, accordingly, is said by intelligent judges to be tedious and embarrassed. We give here the Sanscrit digits, in what are called the *Devanagari* character, and place over them the numeral elements, consisting of a succession of nine simple strokes, variously combined, from which they may, with great probability, be supposed to have been formed; although some Arabian authors, who treat of astrological signs, allege that the Indian numerals were derived simply from the quartering of a circle. The resemblance of those natural marks to the derivative, appears certainly very striking. From the latter, the common Hindu digits, here subjoined, and the vulgar Bengalee, are evidently moulded, with only slight alterations of form. The Birman figures are of the same origin, but have a thin, wirey body, being generally written on the palmyra-leaf with the point of a needle.



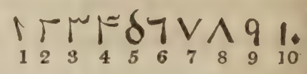
1 2 3 4 5 6 7 8 9 10



1 2 3 4 5 6 7 8 9 10

It appears, from a careful inspection of the manuscripts preserved in the different public libraries of Europe, that the Arabians were not acquainted with the use of the denary numerals, or at least had not generally employed or adopted them, before the beginning of the thirteenth century of the Christian æra. They cultivated the mathematical sciences with ardour, but seldom aspired at original efforts, and generally contented themselves with copying their Grecian masters. The Cufic letters, resembling the Syriac, were introduced a little before the time of Mahomet; but the characters used at present, and called *Neskhi*, was invented about three hundred years afterwards. This alphabet the Arabians employed to express numbers, exactly in the same way as the Greeks. The letters, in their succession, were sometimes applied to signify the lower of the ordinal numbers; but more generally they were distinguished into three classes, each composed of nine characters, corresponding to units, tens, and hundreds. Though, like most of the Oriental nations, the Arabians write from right to left, yet they followed implicitly the Greek mode of ranging the numerals and performing their calculations. For several ages, they hired Christian scribes to keep their accounts; and Walid, Khalif of Syria, prohibited any other marks for numbers to be used than those of the Greeks. With the same deference, they received the other lessons of their great masters, and very seldom hazarded any improvement, unless where industry and patient observation led them incidentally to extend Mensuration, and to rectify and enlarge the basis of Astronomy.

The Arabians might have received their information of the Hindu mode of notation through the medium of the Persians, who spoke a dialect of their language, had embraced the same religion, and were, like them, inflamed by the love of science and the spirit of conquest. The Arabic numerals, here annexed,




though derived apparently from the inversion of the Sanscrit digits, resemble more strikingly the Persic, which are now current over India, and there esteemed the fashionable characters. But the Persians themselves, though no longer sovereigns of Hindostan, yet discover their superiority over the feeble Gentoos, since they generally filled the offices of the revenue, and enjoyed the reputation of being the most expert calculators in the East. It should be observed, however, that, according to Gladwin, these accountants have introduced a peculiar contracted mode of registering very large sums, partly by the numeral characters, and partly by means of symbols formed with abbreviated words, somewhat analogous to the Chaldaic marks.

The Indian origin of the denary numerals is farther confirmed, by the testimony of Maximus Planudes, a monk of Constantinople, who wrote, about the middle of the fourteenth century, a book on Logistics, or Practical Arithmetic, entitled, "*The great Indian Mode of Calculating.*" In the introduction, he explains concisely the use of the characters in notation. But Planudes appears not to have received his information either directly from India, or through the channel of the Persians, the nearest neighbours on the eastern confines of the Greek Empire. It is most probable that he was made acquainted with those numerals by his intercourse with Europe, having twice visited, on a sort of embassy, the Republic of Venice; for, of two manuscripts preserved in the library of St Mark, the one has the characters of the Arabians, and the other has that variety which was first current in Europe, while neither of them shows the original characters used in Hindostan.

It is a more important inquiry to ascertain the period when our present numerals were first spread over Europe. As it certainly preceded the invention of printing, the difficulty of coming to a clear decision is much increased, by the necessity of examining old and often doubtful manuscripts. Some authors would date the introduction of those ciphers as early as the beginning of the eleventh century, while others, with far greater appearance of reason, are disposed to place it two hundred and fifty years later.

While the thickest darkness brooded over the Christian world, the Arabians, reposing after their brilliant conquests, cultivated with assiduity the learning and science of Greece. If they contributed little from their native stores, they yet preserved and fanned the holy fire. Nor did they affect any sort of concealment, but freely communicated to their visitors that precious knowledge which they had so zealously drawn from different quarters. Some of the more aspiring youths in England and France, disgusted with the wretched trifling of the Schools, resorted for information to Spain; and having the courage to subdue the rooted abhorrence entertained in that age against the infidels, took lessons in philosophy from the enlightened Moors. The great objects of study were the *Algarithm* and the *Almagest*, terms derived from the Greek with the Arabic definite article prefixed to them; the former including the rules of Arithmetic, and the latter comprehending the principles of Astronomy.

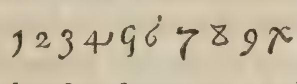
Among the earliest of those who performed such a pilgrimage, was the famous Gerbert, born of obscure parents at Aurillac, in Auvergne, but promoted by his talents, from the condition of a monk, successively to the Bishoprick of Rheims and of Ravenna, and finally elevated to the Papal Chair, which he filled during the last four years of the tenth century, under the name of Sylvester II. This ardent genius studied Arithmetic, Geometry, and Astronomy among the Saracens; and on his return to France, charged with various knowledge, he was esteemed a prodigy of learning by his contemporaries; nor did the ignorance and malice of his clerical brethren fail to represent him as a magician, leagued with the Powers of Darkness. Gerbert wrote largely on Arithmetic and Geometry, and gave rules for shortening the operations with the *Abacus*. In some manuscripts, the numbers are expressed in ciphers; but they had evidently crept in through the licence of transcribers, and it would be most unwarrantable thence to conclude, as many writers have done, that Gerbert had actually the merit of introducing those characters into Europe. The same remark will extend likewise to our celebrated countrymen, Roger Bacon, and John of Halifax, or Holywood, and therefore styled *Sacro-Bosco* in the rude Latinity of that age, who flourished indeed about three centuries later, but must have derived their information, though perhaps not directly, from the same source. Bacon wrote on the reformation of the kalendar, yet he has given no proofs of his acquaintance with the denary notation. Sacro-Bosco composed a treatise on the Sphere, which was long held as a standard work in the Schools. In the latter copies of that book, numeral characters had been sometimes inserted.

The Digital Arithmetic, conjoined with the higher art of Algebra, seems to have been first brought into Europe by the zeal of Leonardo Bonacci, of Pisa, a wealthy merchant who traded to the coast of Africa and the various ports of the Levant. Commercial speculations having tempted him frequently to visit those countries, he was induced, by the love of knowledge, to study thoroughly the science of calculation among the Arabians. On his return to Italy in 1202, this meritorious person composed an Arithmetical Treatise, which he greatly enlarged in 1228. But typography had not yet lent its magic aid to the multiplication of thought, nor do the Tuscans, though long reputed the best calculators in Italy, and consequently in Europe, and to whom we owe the method of Book-keeping, appear to have derived their skill from an acquaintance with the writings of Bonacci. His manuscript had lain more than two centuries neglected, till Lucas Paccioli, or de Burgo, instructed chiefly by its perusal, published, successively between the years 1470 and 1494, the earliest and most extensive printed treatise on Arithmetic and Algebra.

The term *cifer*, or more correctly *sipher*, as appropriated to the digital characters, is an Arabic word introduced by the Saracens into Spain, signifying to *enumerate.* There is little doubt that the Arabic figures were first used by astronomers, and afterwards circulated in the almanacs over Europe. The learned Gerard Vossius places this epoch about the year 1250; but the judicious and most laborious Du Cange thinks that ciphers were unknown before the fourteenth century; and Father Mabillon, whose diplomatic researches are immense, assures us, that he very rarely found them in the dates of any writings prior to the year 1400. Kircher, with some air of probability, seeks to refer the introduction of our numerals to the astronomical tables which, after vast labour and expense, were published by the famous Alphonso, King of Castile, in 1252, and again more correctly four years afterwards. But it is suspected, on very good grounds, that, in the original work, the numbers were expressed by Roman or Saxon letters.

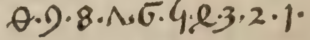
In the *Archæologia*, there is given a short account of an almanac preserved in the library of Bene't College, Cambridge, containing a table of eclipses for the cycle between 1330 to 1348. It has prefixed to it a very brief explication of the use of numerals, and the principles of the denary notation; from which may be seen how imperfectly the practice of those ciphers was yet understood. The figures are of the oldest form, but differ not materially from the present, except that the *four* has a looped shape, and the *five* and *seven* are turned about to the left and to the right. The *one*, *two*, *three*, and *four*, are likewise, perhaps for elucidation, represented by so many dots, thus, . .. ∴ ∷ ; while *five*, *six*, *seven*, and *eight*, are signified by a semicircle or inverted ɔ with the addition of corresponding dots—ɔ ɔ· ɔ : ɔ ∴ *Nine* is denoted by o; *ten* by the same character, with a dash drawn across it; and *twenty*, *thirty*, or *forty*, by this last symbol repeated.

As a farther evidence of the inaccurate conceptions which prevailed respecting the use of the digits in the fourteenth century, we may refer to the mixture of Saxon and Arabic numerals copied from some French manuscripts by Mabillon. The ordinary series is thus express-



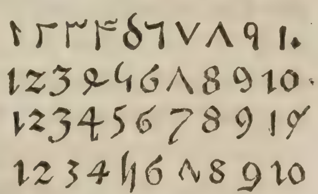
ed, the Saxon ɼ being here employed to denote *ten*, and repeatedly combined with the common digits; nay, ɼɼɼ and ɼɼɼi are immediately followed by 302 and 303, which were therefore intended to signify *thirty-two* and *thirty-three*, the force of the cipher not being yet rightly understood.—It should be observed, that the Greek *Episêmon* or *Fau*, for the number *six*, had come to be represented by a character similar to G.

One of the oldest authentic dates expressed in numeral characters is that of the year 1375, which seems to have been written by the hand of the famous Petrarch on a copy of St Augustin, that had belonged to that distinguished Poet and Philosopher. The use of those characters was only beginning to spread in Europe, and still confined to men of learning. A little tract in the German language, entitled, *De Algorismo*, and bearing the date 1390, explains, with great brevity, the digital notation and the elementary rules of Arithmetic. At the end of a short Missal, similar directions are given in verse, which, from the form of the writing, may be judged to belong to the same period. Here annexed is a correct *fac simile* of the digits themselves, with the modern figures placed below them; but what is very remarkable, the characters in both manuscripts range from right to left, the order which the Arabians would naturally follow.


0, 9, 8, 7, 6, 5, 4, 3, 2, 1,

It was not very easy to comprehend at first the precise force of the *cipher*, which, insignificant by itself, only serves to determine the rank and value of the other digits. A sort of mystery, which has imprinted its trace on language, seemed to hang over the practice of numeration, for we still speak of *deciphering*, and of *writing in cipher*, in allusion to some dark or concealed art. After the digits had come to supply the place of the Roman numerals, a very considerable time probably elapsed before they were generally adopted in calculation. The modern practice of arithmetic remained unknown in England, till about the middle of the sixteenth century; and the lower orders, imitating the clerks of a former age, were still accustomed to reckon by the help of their *awgrym stones.* In Shakespear's comedy of the *Winter's Tale,* written at the commencement of the seventeenth century, the clown, staggered at a very simple multiplication, exclaims that he must try it with counters.

It cannot be doubted, that the kalendars composed in France or Germany, and sent to the different religious houses, were the means of dispersing the knowledge of Arabic numerals over Europe. In the library of the University of Edinburgh, there is a very curious almanac, presented to it, with a number of other valuable tracts, by the celebrated Drummond of Hawthornden, beautifully written on vellum, with most of the figures pencilled in vermillion. It had been calculated particularly for the year 1482, but contains the succession of lunar phases for three cycles, 1475, 1494, and 1513, with the visible eclipses of the sun and moon from 1482 to 1530 inclusive. The date of this precious manuscript, which had once belonged to St Mary's Abbey at Cupar in Angus, is hence easily determined, and the numerals now exhibited were exactly copied from it. For the sake of comparison,



the corresponding Arabic characters are placed above them, and again below them is given a specimen of the current forms which the digits acquired in England, about the middle of the sixteenth century; the last row showing the old figures used by Caxton, when he printed the *Mirrour of the World,* in 1480.

The College accounts in the English Universities were generally kept in the Roman numerals, till the early part of the sixteenth century ; nor, in the parish registers of the South, were the Arabic characters adopted before the year 1600. The oldest date to be met with in Scotland is 1490, which occurs in the rent-roll of the Diocese of St Andrew's; the change from Roman to Arabic numerals, with a corresponding alteration in the form of the writing, appearing near the end of the volume.

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Having endeavoured to trace the origin and introduction of our numeral characters, it only remains now to explain the *operations of Figurate Arithmetic.* But the rules which guide the practice of numbers are easily deduced from the principles already unfolded in treating of *Palpable Arithmetic.* The same theory may likewise suggest other methods of varying and abridging the common operations. I shall follow the same order, selecting as few examples as may be wanted for illustration. The *Denary* Scale, being the one generally received, will claim the chief attention ; but its results shall be compared with those of some other scales, particularly the *Duodenary,* which is partially adopted in commerce, and possesses certain peculiar advantages. For the expressions of this scale, however, it becomes necessary to devise two additional characters to denote *ten* and *eleven.* Not to seek far after such objects, I have contented myself with condensing the ordinary forms into σ and ⊚, which are perhaps sufficiently distinct, while they shadow out the figures represented by them.

The great recommendation of the *Duodenary* scale, consists in its fitness to denote fractional parts. Its index has indeed no fewer than four factors—2, 3, 4, and 6 ; while *ten* is divisible only by 2 and 5. Several attempts, accordingly, have, at different times, been made to carry this scale into actual practice. It is a curious fact, that the famous Charles XII. of Sweden, whose views, though often disturbed by the wildness of heroism, were on the whole beneficent, seriously deliberated on a scheme of introducing this system of numeration into his dominions, a very short time before his death, while lying in the trenches, during the depth of winter, before the towering Norwegian fortress of Frederickshall.

In the *Denary Scale*, for the sake of expedition, it is often convenient to class the digits into members or periods of *three* places, ascending by the powers of a *thousand*. This improvement is due to the Italians, who likewise drew from their elegant dialect the appropriate names for such periods : *units*, *thousands*, *millions*, *billions*, *trillions*, *quadrillions*, *quintillions*, *sextillions*, *septillions*, *octillions*, and *nonillions*. — Thus, the number 5,482,584,878,284,800, which expresses the square feet in the surface of our globe, is read, 5 *quadrillions*, 482 *trillions*, 584 *billions*, 878 *millions*, 284 *thousands*, and 800 *units*.

In many cases, it would facilitate calculation, to have figures corresponding to open counters. To transform the ordinary characters, therefore, into deficient digits, I have caused modify their shape thus :

1 2 3 4 5 6 7 8 9 0

an alteration sufficient to distinguish, without entirely altering, or disguising, them. With such reclined figures, it will be easy to represent numbers, by their defects as well as their excesses. This answers most conveniently in expressing the digits from 6 to 9 inclusive. Thus 38 may be denoted by 42, meaning 40 with 2 abated ; for the same reason, 829 may be written 1231, signifying that 1030 is to be diminished by 201.

## NUMERATION

Being only the mode of classing numbers by successive braces, leashes, warps, &c. is performed, by dividing them continually by the root of the scale of arrangement. In the reduction, consequently, to the *Binary*, *Ternary*, *Quaternary*, &c. system, the corresponding divisor is *two*, *three*, *four*, &c. As an illustration, let it be required to exhibit the number 430685 on a variety of scales. The decomposition will be thus effected :

**BINARY.**

| | |
|---|---|
| 2)430685 | |
| 215342 | 1 |
| 107671 | 0 |
| 53835 | 1 |
| 26917 | 1 |
| 13458 | 1 |
| 6729 | 0 |
| 3364 | 1 |
| 1682 | 0 |
| 841 | 0 |
| 420 | 1 |
| 210 | 0 |
| 105 | 0 |
| 52 | 1 |
| 26 | 0 |
| 13 | 0 |
| 6 | 1 |
| 3 | 0 |
| 1 | 1 |

**TERNARY.**

| | |
|---|---|
| 3)430685 | |
| 143561 | 2 |
| 47853 | 2 |
| 15951 | 0 |
| 5317 | 0 |
| 1772 | 1 |
| 590 | 2 |
| 196 | 2 |
| 65 | 1 |
| 21 | 2 |
| 7 | 0 |
| 2 | 1 |

**QUATERNARY.**

| | |
|---|---|
| 4)430685 | |
| 107671 | 1 |
| 26917 | 3 |
| 6729 | 1 |
| 1682 | 1 |
| 420 | 2 |
| 105 | 0 |
| 26 | 1 |
| 6 | 2 |
| 1 | 2 |

**QUINARY.**

| | |
|---|---|
| 5)430685 | |
| 86137 | 0 |
| 17227 | 2 |
| 3445 | 2 |
| 689 | 0 |
| 137 | 4 |
| 27 | 2 |
| 5 | 2 |
| 1 | 0 |

**SENARY.**

| | |
|---|---|
| 6)430685 | |
| 71780 | 5 |
| 11963 | 2 |
| 1993 | 5 |
| 332 | 1 |
| 55 | 2 |
| 9 | 1 |
| 1 | 3 |

**SEPTENARY.**

| | |
|---|---|
| 7)430685 | |
| 61526 | 3 |
| 8789 | 3 |
| 1255 | 4 |
| 179 | 2 |
| 25 | 4 |
| 3 | 4 |

**OCTARY.**

| | |
|---|---|
| 8)430685 | |
| 53835 | 5 |
| 6729 | 3 |
| 841 | 1 |
| 105 | 1 |
| 13 | 1 |
| 1 | 5 |

**NONARY.**

| | |
|---|---|
| 9)430685 | |
| 47853 | 8 |
| 5317 | 0 |
| 590 | 7 |
| 65 | 5 |
| 7 | 2 |

**UNDENARY.**

| | |
|---|---|
| 11)430685 | |
| 39153 | 2 |
| 3559 | 4 |
| 323 | 6 |
| 29 | 4 |
| 2 | 7 |

**DUODENARY.**

| | |
|---|---|
| 12)430685 | |
| 35890 | 5 |
| 2990 | 0 |
| 249 | 2 |
| 20 | 9 |
| 1 | |

Hence the number 430685 will be thus represented on different scales:

| | | | |
|---|---|---|---|
| Binary, ...... | 1101001001001011101 | Octary, ...... | 1511135 |
| Ternary, .... | 210212210022 | Nonary, ..... | 725708 |
| Quaternary, | 1221021131 | Denary, ..... | 430685 |
| Quinary, ...... | 102240220 | Undenary, .. | 274642 |
| Senary, ........ | 13121525 | Duodenary, | 1892$\phi$5 |
| Septenary, ... | 3442433 | | |

The notation may be readily transferred to any higher scale, which has for its index some power of that of the lower. Let the given number be distinguished into periods consisting of two, three, or four places; the amount of these may be condensed into a single figure, and the corresponding index then is the second, third, or fourth power of the primary index. Thus, conceive the expression 1, 10, 10, 01, 00, 10, 01, 01, 11, 01, of the *Binary Scale*, to articulate at every alternate place from the right hand, the value of each period, or 1 for 01, 3 for 11, and 2 for 10, will transform the whole into this *Quaternary* arrangement, 1221021131. If the same expression be distinguished into triplets, 1, 101, 001, 001, 001, 011, 101, and each of these afterwards compressed into a single figure, assuming 5 for 101, and 3 for 011, it will be changed into the *Octary* notation 1511135.—In like manner, if the representation of the same number on the *Ternary Scale* be broken thus, 21, 02, 12, 21, 22, 00, 22, at every alternate place, and the values of those periods adopted, 8 being substituted for 22, 7 for 21, and 5 for 12, the whole will be converted into the expression 725708 of the *Nonary Scale*.

To transfer, in general, any numeral expression from one scale to another, the most obvious way is to decompound it, and then dispose it again on the new scale. Suppose it were required to convert the *Ternary expression* 210112 into the *Quinary Scale:* Beginning at the left hand, and constantly *tripling* the terms, shifting them every time a place lower, the successive collected results are 2, 7, 64, 193, and 581. This final number, if decomposed again in the ordinary mode by a series of *pentads*, would give 4311.

But this conversion might be performed directly with more expedition, by dividing the original expression and the quotients successively which arise, by the index of the new scale, as exhibited in the same notation. Thus, let it be required to transform the *Quinary* expression 102240220 to the *Octary Scale.* It is only necessary to observe, that the operation is carried on through the *Quinary Scale*, or by a series of *pentads.* In the first division, 102 makes *twenty-seven*, which contains the index *three* times with a remainder of 3; then 30 is equivalent to *seventeen*, which contains it only *twice* with an excess of 1; and 14 or *nine* contains it only *once*, with an excess likewise of 1. In this manner, the division is pursued continually, till the decomposition becomes completed. The result is, therefore, 1511135, the same as formerly stated.

| | |
|---|---|
| 8)102240220 | |
| 32210320 | 5 |
| 203404 | 3 |
| 11331 | 1 |
| 410 | 1 |
| 23 | 1 |
| 1 | 5 |

As another example, suppose it were sought to convert the *Nonary* expression 725708 into a *Senary* one. The constant divisor is now 6, but the operation is performed on the *Nonary Scale*, the values of the digits advancing always by *nines.* Here 6 is contained *once* in 7, and *once* again in 12 or *eleven*, but *eight* times in 55 or *fifty.* The rest of the operation proceeds in like manner. The equivalent *Senary* expression is hence 13121525.

| | |
|---|---|
| 6)725708 | |
| 118415 | 5 |
| 17362 | 2 |
| 2654 | 5 |
| 408 | 1 |
| 61 | 2 |
| 9 | 1 |
| 1 | 3 |

Lastly, to convert a *Septenary* into an *Undenary* expression, the process would be thus carried on. The index *eleven* being denoted on the *Undenary Scale* by 14, this number forms the divisor; but 14 is contained in 34 *twice*, with a remainder of 3 on the *Septenary Scale*; and the next dividends are 34, 32, 14, &c. The remainder of the final division is 10, that is *seven*, on the scale of operation. Whence the corresponding *Undenary* expression is 274642.

```
14)3442433
    222102 | 2
     13243 | 4
       641 | 6
        41 | 4
         2 | 10
```

From the principle before investigated in Palpable Arithmetic, we can easily find the remainder of the division of the original number by another number, which is one less than the index of any particular scale. Thus, to begin with the *Ternary Scale*, if the amount of all the figures be divided by *two*; or, what is the same thing, if every *two* be rejected, and the other figures successively added, retaining only, at each step, the excess above *two*; at the end of this operation, there will be left *one*. Hence it might be concluded that 430685 is an odd number; a property indicated also by the character of its last digit.—In the *Quaternary Scale*, by adding the figures together, and constantly throwing out the *threes*, there remain *two*; which shows that the division of the original number by *three* would leave *two*.—In the *Quinary Scale*, the several figures being collected, omitting the *fours* as they arise, give *one*, for the remainder of a division of 430685 by *four*.—In the *Senary Scale*, omitting all the *fives*, there is no remainder; a proof that the given number is divisible by *five*.—In the *Septenary Scale*, collecting the figures and rejecting the *sixes*, there is an excess of *five*, intimating *five* as the remainder of the division by *six*.—In the *Octary Scale*, rejecting the *sevens*, there are left *three*, being the excess of the division by *seven*.—In the *Nonary Scale*, omitting the *eights*, the surplus is *five*, corresponding to the remainder of the division by *eight*.—In the *Denary Scale*, by adding the figures, and rejecting the *nines* as fast as they arise, there is still an excess of *eight*, being the remainder of the division of 430685 by *nine*.—In the *Undenary Scale*, by casting out the *tens*, there is left *five*, or the last digit of the original number, and consequently the remainder of its division by *ten*.—Finally, in the *Duodenary Scale*, by separating the *elevens* from the collected figures, there remain *two*, the last figure in the preceding scale, or the residue of the division by *eleven*.

In general, the terminating figure of the expression on any scale, is the same as the remainder of the division by its index on the next higher scale. The casting out of the divisor may be shortened likewise, if it be contained in any power of the index, by taking only the corresponding terms. Thus, not to go farther than the *Denary Scale*, *two* may be cast out merely from the division of the final digit, or 5. *Four* may be separated by dividing the two last figures, or 85; and the remainder of the division by *eight* is detected, from the examination of the three terminating digits 685.

But the quotients of such divisions are also directly discovered, from what was explained under *Palpable Arithmetic*. Not to multiply examples, let us begin with the *Senary Scale*. If the several figures 13121525 be repeated on all the lower places, their summation will appear as here annexed. The excesses corresponding to the different rows amount to 20, which, divided by 5, leaves no remainder, but gives 4 to the next column. The sum of this again, with the 4 carried to it, is 19, which contains 6 *three* times, with an excess of 1. This 3 is conveyed to the next column, and the same operation is repeated. The result of the whole, or 1502441, being now transferred from the *Senary* to the *Denary Scale*, gives 86137, for the quotient of 430685 by *five*.

| | |
|---:|:---|
| 1111111 | 1 |
| 333333 | 3 |
| 11111 | 1 |
| 2222 | 2 |
| 111 | 1 |
| 55 | 5 |
| 2 | 2 |
| | 5 |
| 1502440 | 0 |

The expression of the *Octary Scale*, treated in the same way, will stand thus. Here the excesses taken together make up 17, giving, after the division by 7, a surplus of 3, and 2 to be carried to the first column. The figures on all these columns, being collected and reckoned by *eights*, amount to 170126, which, converted again to the *Denary Scale*, is 61526, or the quotient of the original number by *seven*, with a remainder of 3.

| | |
|---:|:---|
| 111111 | 1 |
| 55555 | 5 |
| 1111 | 1 |
| 111 | 1 |
| 11 | 1 |
| 3 | 3 |
| | 5 |
| 170126 | 3 |

On the *Denary Scale* itself, the same decomposition is here pursued. The several excluded figures now amount to 26, making 2 *nines*, and a surplus of 8. The columns themselves being summed up, give 47853, for the quotient by *nine*, with a remainder 8.

| | |
|---:|:---|
| 44444 | 4 |
| 3333 | 3 |
| 66 | 6 |
| 8 | 8 |
| | 5 |
| 47853 | 8 |

Lastly, suppose the expression of the *Duodenary Scale* were thus analysed: The excesses amount to 35, or 3 *elevens* and 2. The columns added in succession give 1σ7σ9, which, converted into the ordinary scale, makes 39153, with a surplus of 2, for the quotient of 430685 by *eleven*.

| | |
|---:|:---|
| 11111 | 1 |
| 8888 | 8 |
| 999 | 9 |
| 22 | 2 |
| σ | σ |
| | 5 |
| 1σ9σ9 | 2 |

But the extension which was made of the same principle to open counters, may be applied, to discover the quotient of any number divided by the index of the next higher scale. Without dwelling on this subject, we shall take the three last examples to illustrate the mode of operation. Beginning, therefore, at the left hand, each figure must be repeated through all the succeeding places, alternately as an excess and a defect, and the balance of their addition, or 143436, taken as the result. This number is otherwise expressed 135356, which corresponds to 47854, the quotient of the original number, with the accession of the excluded 1, by *nine*.

| | |
|---|---|
| 1\1\1\ | 1 |
| 55555 | 5 |
| 1\1\ | 1 |
| 1\1 | \ |
| 1\ | 1 |
| 3 | 3 |
| | 5 |
| 143436 | \ |

Again, the analysis will proceed thus in the ordinary scale. The extended digits give a surplus of 2, and the figures on the several columns amount to 4\153, or 39153, being the quotient of 430685 by *eleven*, with an excess of 2.

| | |
|---|---|
| 44444 | 4 |
| 3333 | 3 |
| 66 | 6 |
| 8 | 8 |
| | 5 |
| 4\153 | 2 |

Lastly, in the *Duodenary Scale*, the decomposition will be more rapid. The surplus figures here give a defect of 5, and the rest in the several columns make up 17200, which is equivalent to 33130, the quotient of the original number, with the addition of 5 by *eleven*.

| | |
|---|---|
| 1\1\1 | \ |
| 8888 | 8 |
| 999 | 9 |
| 22 | 2 |
| 0 | 0 |
| | 5 |
| 17200 | 5 |

It is of more consequence, however, in such divisions, to discover the remainder than the quotient. If the process be confined to the *Denary Scale*, the number *eleven* will appear to have properties analogous to those of *nine*. But to find the remainder of the division of any number by *eleven*, or, in the vulgar phrase, *to cast out the elevens*, will require attention to the alternate character of the ciphers, fluctuating in succession from excess to defect. The easiest mode is, *Beginning at the right hand, to mark the alternate figures; and, from the amount of these, augmented by eleven, if necessary, take that of the rest, and the difference is the remainder sought.* Thus, resuming the original number 43′06′85′, the sum of the accented figures is 14, and that of the rest only 12; wherefore, if divided by *eleven*, it would leave an excess of 2. Again, taking the number 3′17′06′25′, the marked figures amount to 21, and the others only make 3, leaving for the division by *eleven* a deficiency of 18 or 7, that is, an excess of 4.

It hence follows, that, as a number is divisible by *nine*, when the amount of its figures is any multiple of *nine*; so a number is divisible by *eleven*, when the sums of the alternate figures are either equal, or differ by *eleven* or its multiples. This proposition leads to some curious results, but I shall notice only the more striking and simple. It is an obvious consequence, that the *difference* between any number and its reverse is always divisible by *nine*: Thus, the number 430685 being reversed into 586034, gives the difference 155349, which may be divided without any remainder, by 9. The reason is plain, since this number and its reverse are expressed by identical figures, they are both multiples of 9 with the same excess, and consequently their difference must only be some multiple of 9.—Again, the *difference* between a number and its reverse is likewise divisible by *eleven*, if it has *odd* places of figures. Thus, the difference between 3170625 and its reverse, or 2090088, is divisible by *eleven*; for the sums of the alternate figures, 19 and 8, differ by 11. But the *sum* of a number and its reverse is divisible by *eleven*, when it consists of *even* figures. Thus, the original number 430685, having 586034 for its reverse, their sum is 1′01′67′19′; which is evidently divisible by *eleven*, since the accented figures amount to 18, while the rest make only 7, or 11 less.

It is not difficult to perceive the reason of these properties of *eleven*. When the number consists of odd figures, they preserve the same character of abundant or defective in its reverse, and consequently the *subtraction* of the opposite numbers will destroy whatever inequality there had before existed; but when the number proposed consists of even figures, the abundant and defective, by reverting their order, mutually change places, and hence the *addition* of the number and its reverse will extinguish any original inequality between these, counterbalancing any surplus of the one set by an equal deficiency in the other.

The number *seven* is likewise distinguished by its properties on the *Denary Scale*, though they are not quite so remarkable as the relations of *nine* and *eleven*. All the remainders of any division by 7 must evidently be included in 1, 2, 3, 4, 5, or 6. But if *units* having ciphers annexed in succession be divided by 7, the quotient is found to be 142857, with the corresponding remainders 1, 3, 2, 6, 4, 5, which comprehend all the possible varieties. Wherefore, since the last of these remainders is just as it was at first, the series of divisions will again be renewed; and consequently, in the expression of the quotient, the same digits must perpetually recur, and in the same order. Nay, if the first period of that quotient, or the number 142857, be multiplied by 2, 3, 4, 5, or 6, the products, 285714, 428571, 571428, 714285, or 857142, are still denoted by the same digits, and in the same order, only commencing from different points. The reason of this very curious property is, that the remainders 1, 3, 2, 6, 4, 5, precede the digits, 4, 2, 8, 5, and 7, of the quotient; or, in other words, those remainders with ciphers annexed, when divided by 7, give quotients commencing with such digits, and afterwards running through all the series of changes. Now, it is evident, that these new quotients must be the same as the first one multiplied by the several remainders.

Another remarkable property of the quotients by 7, which may be of some utility in Practical Arithmetic, is derived from similar principles. Since the remainders of the division of 1, 10, 100, 1000, &c. were respectively 1, 3, 2, 6, 4, and 5, it follows that the remainder of the division of any of these digits, with annexed ciphers by 7, will be found by pursuing the same concatenated order, 1, 3, 2, 6, 4, 5; 1, 3, 2, 6, 4, 5, &c. and reckoning downwards from the given digit to the last place, or that of units. Thus, the remainder of the division of 400 by 7, is 1, because 1 stands in the series *two* places lower than 4, which answered to hundreds; for the same reason, 5 is the remainder of the division of 2000, since it occurs *three* places lower than 2. If a larger digit than 6, such as 8 or 9, with its train of ciphers, be divided by 7, the remainder will evidently be the same as what corresponds to the excess of 1 or 2. It will be hence easy to discover the remainder of the division of any compound number by 7. Suppose, for example, the former number 430685 were proposed: Beginning at the right hand, and conceiving it to be composed of 5, 80, 600, &c. the corresponding remainders are 5, 3, 5, 5 and 6, making up 24, or a general excess of 3.

In Palpable Arithmetic, the primary bar was always marked, as the key to ascertain the import of the rest. The same thing is done in the digital notation, by setting a full point immediately after the place of units. Below that point, the digits must diminish exactly as they increase in standing above it. The descending terms of any numerical scale are hence fitted to denote the remotest subdivision of parts with equal facility, as the ascending ones are capable of expressing the largest possible number. This property, though but slow in being perceived, constitutes one of the greatest advantages of such scales.

Hence to transfer the lower denominations from one scale to another, it is only required to multiply continually the given expression by the index of the new scale, and conceive the product each time to be set a place lower. Thus, to change the fractional digits .15243134, denoting the diameter of a circle which has unit for its circumference, from the *Senary* to the *Quaternary Scale*. This repeated multiplication is performed after the *Senary notation;* the products of the single digits being constantly divided by *six*, while the remainder is set down, and the quotient carried to the next place. The corresponding *Quaternary notation* is therefore, in round numbers, .11011331, the same as what arises from the decimal expression .3810938062.

| |
|---:|
| .15243134 |
| 4 |
| 1.13501024 |
| 4 |
| 1.03204144 |
| 4 |
| 0.21225104 |
| 4 |
| 1.25352424 |
| 4 |
| 1.54334544 |
| 4 |
| 3.50231504 |
| 4 |
| 3.21411224 |
| 4 |
| 1.30445344 |

The descending scales now selected for explaining the method of transformation are seldom ever used. But the descending terms of the *Denary Scale*, or those of the Decimal Subdivision, have at length been very generally adopted into practice, at least in all the calculations connected with mathematical and physical science. The duodecimal system of partition, from its convenience in Mensuration, is employed to a certain extent among traders. It may be proper, therefore, to exemplify the reciprocal transformation of decimals and duodecimals. In this view, let it be required to change the expression .785398, which denotes the ratio of the circle to its circumscribing square, into duodecimals. The operation is performed by multiplying the given fraction by 12, and pointing off the same number of digits; the process being constantly repeated with the several partial excesses. Hence the equivalent expression in the duodecimal notation is .951201σ; of which the four first digits, however, may be judged sufficient in practice.

.785398<br>12<br>9.424776<br>12<br>5.097312<br>12<br>1.167744<br>12<br>2.012928<br>12<br>0.155136<br>12<br>1.861632<br>12<br>σ.339584

Again, the cube root of 2 expressed in duodecimals is 1.315188. To reduce this to decimals, it must undergo a repeated multiplication by *ten* or σ. It is scarcely necessary to observe, that the notation being now duodenary, the whole process is to be carried through that scale, each digital product which arises in succession being continually reckoned by *twelves*. Thus, *ten* times 8, or 80, is *six* dozen and 8 over; and the next product 80 with the 6 carried, is *seven* dozen and 2 over. In like manner is the operation performed with all the other figures. The result of the conversion is hence, in decimals, 1.259921.

1.315188<br>σ<br>2.723528<br>σ<br>5.00σ428<br>σ<br>9.007628<br>σ<br>9.263228<br>σ<br>2.127σ28<br>σ<br>1.026628

Both these last descending scales, however simple and commodious in practice, are comparatively of recent adoption. The natives of India, who have so long been acquainted with the *Denary Notation*, are still ignorant of its application to fractions. Below the place of units, they change the rate of progression, and descend merely by a continued bisection, assuming successively the *half*, the *fourth*, the *eighth*, and the *sixteenth*; and beyond this last partition they seldom advance. Nor did the Moors, during the short period in which they cultivated science after receiving the digital system from the East, ever attain to the knowledge of *Decimal Arithmetic*. All progress was fatally stopped by their cruel expulsion from Spain; but, that the principle of the *Denary System* should, in its native bed, have lain absolutely unproductive through the course of many centuries, is a circumstance which strongly marks the want of invention among the Hindus.

It is curious to remark, that the use of decimal fractions in calculation was long preceded by the complex train of *Sexagesimals*. This rapid progression had been introduced into the Alexandrian school by the famous Ptolemy, who had the merit of digesting the results of astronomical observations into a body of regular science. It descended by the powers of *sixty*; but though quite artificial and seemingly arbitrary in its structure, it was easily engrafted on the Greek system of numeration. The astronomers of Alexandria and of Constantinople continued to employ the *Sexagesimal Notation*, in which they were afterwards imitated by their successors among the Arabians and Persians. The system itself had no doubt its rise in the subdivisions suggested by the celestial phenomena. The partition of the circumference of the circle into 360 equal degrees was originally founded on the supposed length of the year, which, expressed in round numbers, consists of twelve months, each composed of thirty days. The radius, approaching to the sixth part of the circumference, would contain nearly 60 of those degrees; and after its ratio to the circumference was more accurately determined, the radius still continued to be distinguished into the same number of divisions, which likewise bore the same name. As calculation became more refined, each of these 60 divisions of the radius was, following the uniform progression, again subdivided into 60 equal portions, called minutes or primes; and, by repeating the process of sexagesimal subdivision, seconds and thirds were successively formed. The degrees were considered as integers, and the minutes, or primes, the seconds, and thirds, &c. distinguished by intervening blanks, and sometimes the addition of corresponding dashes.

As an illustration of the mode of converting decimals into sexagesimals, let it be required to express the side of an inscribed decagon, or the chord of 36°, in *sexagesimal* parts of the radius. Here the decimal quantity, .61803398428, denoting the greater segment of this line considered as *unit*, and divided into extreme and mean ratio, is multiplied by 60, or, what is equivalent, by 6, with one decimal abridged each time, the same process being repeated on all the successive fractions. The result is consequently $37^\circ\ 4'\ 55''\ 20'''$ $26^{iv}\ 10^{v}\ 34^{vi}$; which exactly corresponds, as far as the seconds, with what Ptolemy

$$
\begin{array}{r}
.61803398428 \\
60 \\
\hline
37.0820390568 \\
60 \\
\hline
4.922343408 \\
60 \\
\hline
55.34060448 \\
60 \\
\hline
20.4362688 \\
60 \\
\hline
26.176128 \\
60 \\
\hline
10.56768 \\
60 \\
\hline
34.0608
\end{array}
$$

has assigned. He stops there, and the accuracy now pursued for the sake of exemplification, is indeed superfluous, approaching within a *trillionth* part of the truth.

Another example will show, by the converse procedure, that the Greek astronomer knew more accurately, than has generally been supposed, the length of the circumference of the circle. He calculates the chord of *one* degree, which cannot differ sensibly from the arc itself, to be 1° 2′ 50″. Consequently, if the radius were considered as *unit*, the arc of 60 degrees would be represented sexagesimally by 1.2′ 50″; and therefore the triple of this, or 3.8′ 30″, must express the circumference of a circle whose diameter is 1. To reduce that quantity to decimals, the fractional parts are multiplied repeatedly by 10, and the successive products divided by 60. Hence the decimal expression for the circumference of a circle is 3.1416, a very useful and celebrated approximation.

$$\begin{array}{r}
3.\ 8'\,30'' \\
10 \\
\hline
1.25\ 00 \\
10 \\
\hline
4.10 \\
10 \\
\hline
1.40 \\
10 \\
\hline
6.40
\end{array}$$

It was the practice of sexagesimals, at a late period, that led by gradual steps to the formation of Decimal Arithmetic. The great restorer of mathematical science in Europe, George Purbach, or Beurbach, of Vienna, a man of original and extensive genius, who died at an early age in 1462, in the table of sines which he appears to have computed to every minute of the quadrant, instead of distinguishing the radius into 216,000 seconds, or dividing it three times in succession by 60, made it to consist first of 600 instead of 60 equal portions, and then parted each of these into 100 primes, and each prime again into 100 seconds, thus blending in effect the sexagesimal with the decimal or centesimal notation. His disciple and successor, John Müller, commonly styled Regiomontanus, from Koningsberg the place of his birth, extended those tables to seven places of figures, making the radius to consist of 6,000,000 parts. After some hesitation, he finally abandoned that radical division, and having in 1464 enlarged the radius to *ten million* of parts, he recalculated the sines, to which he likewise joined, for the first time, a table of tangents. But this laborious and important work lay, many years after the author's death, in manuscript, and did not appear before the public until 1541, when it was printed under the direction of Schöner at Nuremberg.

It is obvious that those mixed sexagesimal expressions would be reduced to common decimals, by multiplying by 10 and dividing the product by 6. Thus, in Müller's first table, the sine of a *Kardaga*, or the arc of 15°, so called, it would seem, from the Arabic verb *Karatha*, *to divide*, is 1552914. Wherefore the result of this reduction, with the decimal point prefixed, is .2588190, which perfectly agrees with our modern tables.

$$\begin{array}{r} 1552914 \\ 10 \\ \hline 6)\overline{15529140} \\ \hline 2588190 \end{array}$$

But the final step towards the use of decimals, which consisted in estimating the radius as *unit* successively decomposed, was slowly attained, and a long period still elapsed before mathematicians were trained to the new practice. It is curious to observe here the very gradual progress of improvement. Ptolemy had distinguished the sexagesimal subdivision into primes, seconds, thirds, &c. by corresponding accents placed over the successive parts. Michael Stifelius of Eslingen, the scholar and follower of Luther, having remarked the relations of arithmetical and geometrical progressions, in his *Arithmetica Integra*, printed at Nuremberg in 1545, a work of great merit, noted the exponents of powers, both ascending and descending, by the digits 1, 2, 3, 4, &c. Guided by analogy, he likewise appropriated the zero to indicate unit, or the commencement of every series. He therefore expressed integers and their sexagesimals by these characters, 0, 1, 2, 3, 4, &c. placed over them; and in representing astronomical quantities or physical subdivisions, he combined the exponents with certain contractions or modified accents; thus, $\tilde{1}$, $\tilde{2}$, $\tilde{3}$, $\tilde{4}$, &c. Bombelli, in his Algebra, printed at Bologna in 1572, adopted this improvement. Simon Stevinus of Bruges, mathematician to the first Prince of Orange, and a man of great originality of conception, in his *Arithmetic*, composed in 1583, in the Flemish dialect, and published two years afterwards in French, employed the marks 3, 2, 1, 0, 1, 2, 3, 4, or the digits circumscribed by a circle, to signify the extended series of powers. This notation he applied principally to the *Denary Scale*, setting out from the place of units, and marking the tens, hundreds, thousands, &c. upwards, and again downwards, the tenths, the hundredths, the thousandths, &c. by continual decimation. The units themselves he indicated by 0, implying only the commencement of numeration. Thus, according to him, $\overset{0}{3}\ \overset{1}{1}\ \overset{2}{4}\ \overset{3}{1}\ \overset{4}{6}$ would denote 3 *units*, 1 *tenth*, 4 *hundredths*, 1 *thousandth*, and 6 *ten thousandths*. He first explained the use of such decimals, and strongly urged their preference to vulgar fractions. Still, however, this notation, though sufficiently clear, might seem rather overloaded.

It was not distinctly or immediately perceived, that the mark 0 might, by its position alone, ascertain the rest.

The ultimate simplification, therefore, consisted in omitting altogether those various marks, and placing a full point or a comma, to represent the key, after the units and before the range of decimals. This capital step, which indeed leaves nothing more to be done in practice, we owe most probably to the great Napier, who, in his *Rabdologia*, printed at Edinburgh in 1617, while he quotes Stevinus with applause, incidentally proposes the final improvement. He seems not to trust exclusively, however, to punctuation, and while he marks the termination of the units by a comma, he also cautiously notes the successive decimals by superscribing repeated accents. But the noble invention of logarithms, deriving its birth in like manner from the efforts made to abridge the operations of spherical trigonometry, gave a decided preponderance to decimals, which those artificial numbers, as next remodelled in the hands of Briggs, adopted into their actual composition.

To reduce vulgar fractions to any scale, we have only to multiply the numerator by the root of that scale, and divide by the denominator; and to repeat this process, if requisite, on the successive remainders, till the quotients either terminate absolutely, or glide into a circulation. Suppose it were sought to represent on the *Senary*, *Octary*, and *Denary Scales*, the fraction $\frac{355}{113}$ or $3\frac{16}{113}$, which Peter Metius, a distinguished Dutch mathematician, and near relation of Adrian Metius of Alkmaer, about the close of the sixteenth century, assigned for the approximative ratio of the circumference to the diameter of a circle. The numerator 16 must, therefore, be multiplied by the index of each scale, and the product divided by the denominator 113; the remainder to be treated in the same way in repeated succession. The expression of $\frac{355}{113}$ on the *Senary Scale* is, therefore, 3.0503301; on the *Octary Scale*, 3.110376; and on the *Denary Scale*, 3.141593.

| SENARY. | OCTARY. | DENARY. |
|---|---|---|
| 16 | 16 | 16 |
| 6 | 8 | 10 |
| 113)96(0 | 113)128(1 | 113)160(1 |
| 6 | 113 | 113 |
| 576(5 | 15 | 47 |
| 565 | 8 | 10 |
| 11 | 120(1 | 470(4 |
| 6 | 113 | 452 |
| 66(0 | 7 | 18 |
| 6 | 8 | 10 |
| 396(3 | 56(0 | 180(1 |
| 339 | 8 | 113 |
| 57 | 448(3 | 67 |
| 6 | 339 | 10 |
| 342(3 | 109 | 670(5 |
| 339 | 8 | 565 |
| 3 | 872(7 | 105 |
| 6 | 791 | 10 |
| 18(0 | 81 | 1050(9 |
| 6 | 8 | 1017 |
| 108(1 | 648(6 | 33 |
| 113 | 678 | 10 |
| | | 330(3 |
| | | 339 |

This last conversion would evidently take a more compact form, as here exhibited. The several steps of the process are virtually the same as before. Hence the foundation of the ordinary process for changing a vulgar into a decimal fraction, by dividing the numerator, with ciphers annexed to it, by the denominator.

```
113(16.000000).141593
    113 · · · · ·
     470
     452
      180
      113
       670
       565
       1050
       1017
         330
         339
```

Mixed fractions are reduced to any scale nearly in the same way. Suppose it were sought to express decimally *fifteen shillings* and *threepence three farthings*. Multiply this sum by 10, and the fractional part of the product again repeatedly; the integral results being deferred to the lower places, must evidently express the same value. The decimal corresponding to the mixed fraction is hence .765625.

$$\begin{array}{r}
15\ \ 3\tfrac{3}{4} \\
\underline{10} \\
7.13\ \ 1\tfrac{1}{2} \\
\underline{10} \\
6.11\ \ 3 \\
\underline{10} \\
5.12\ \ 6 \\
\underline{10} \\
6.5 \\
\underline{10} \\
2.10 \\
\underline{10} \\
5.0
\end{array}$$

The same result might have been obtained otherwise, by ascending progressively from the lowest term. Thus, three farthings, expressed in decimal parts of a penny, are .75; and dividing 3.75 by 12, the quotient .3125 denotes the value, with threepence annexed, in decimal parts of a shilling; and finally, having prefixed the fifteen shillings, and divided the compound 15.3125 by 20, the quotient, expressing the fraction of a pound, is the same as before.

But examples of this kind are better adapted for the *Duodenary Scale*. The multiplication here being performed by successive *twelves*, the *Duodecimal* expression for the same sum is therefore .923, consisting only of three figures.

$$\begin{array}{r}
15\ \ 3\tfrac{3}{4} \\
\underline{12} \\
9.\ 3\ \ 9 \\
\underline{12} \\
2.\ 5\ \ 0 \\
\underline{12} \\
3.\ 0
\end{array}$$

As another example of the conversion of mixed fractions, let it be required to find the decimal of a *Ton* corresponding to *thirteen hundred weight, two quarters,* and *seven pounds*. The operation is performed in two ways, which give the same result, .678125. The ascending process, which has 28, 4 and 20 for successive divisors, is perhaps the easier.

$$\begin{array}{llr}
13\text{ cwt.} & 2\text{ q.} & 7\text{ lb} \\
 & & \underline{10} \\
6.15 & 2 & 14 \\
 & & \underline{10} \\
7.16 & 1 & 0 \\
 & \underline{10} & \\
8.\ 2 & 2 & \\
 & \underline{10} & \\
1.\ 5 & 0 & \\
\underline{10} & & \\
2.\ 10 & & \\
\underline{10} & & \\
5.\ 0 & &
\end{array}
\qquad
\begin{array}{r}
28)7.00(.25 \\
\underline{56} \\
140 \\
\underline{140} \\
\\
4)\underline{2.2500} \\
.5625 \\
\\
20)\underline{13.562500} \\
.678125
\end{array}$$

Having explained so fully the principles of Numeration, we now proceed to treat of the common operations in Figurate Arithmetic.

---

## ADDITION.

From the principle of numerical notation, it follows that Addition is performed by collecting the digits of each bar or rank. Each class, whether it be units, hundreds or thousands, is treated in the same way. In adding two figures, it is only requisite to count forwards from one of them, as many steps as are signified by the other. Suppose 5 were to be joined to 8; reckoning onwards, we pass through 9, 10, 11, 12, to 13. This simple process may be more conveniently performed by counting over the fingers. But, for a learner, it is a preferable mode to frame a *Table of Addition*, which he may easily commit to memory. The construction and use of such a table are so very simple, as hardly to require any explanation. The one number occupies the horizontal row at the top, and the other, which is not greater, the vertical row at the side. Thus, below the column of 7, and opposite to the horizontal range of 6, stands 13, the sum of these numbers. Such tables are found in the more ancient treatises of arithmetic; but they have been most injudiciously omitted in the latter systems of education.

*Denary Scale.*

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 1 |
| | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 2 |
| | | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 3 |
| | | | 8 | 9 | 10 | 11 | 12 | 13 | 4 |
| | | | | 10 | 11 | 12 | 13 | 14 | 5 |
| | | | | | 12 | 13 | 14 | 15 | 6 |
| | | | | | | 14 | 15 | 16 | 7 |
| | | | | | | | 16 | 17 | 8 |
| | | | | | | | | 18 | 9 |

Let it be sought to add these four numbers, 3709, 8540, 2618, and 706. Having set them in their ranks, the most natural way would be to write down the sum of each column. The first column on the right hand gives 23, the next 5, the third 25, and the last column 13. Or the operation might be carried on from the left hand to the right, the several sums being now 13, 25, 5, and 23. These numbers, collected by a second summation, give for the final result 15573.

| | |
|---|---|
| 3709 | 3709 |
| 8540 | 8540 |
| 2618 | 2618 |
| 706 | 706 |
| 13 | 23 |
| 25 | 5 |
| 5 | 25 |
| 23 | 13 |
| 15573 | 15573 |

The former mode of working is generally preferred in Europe, but the latter is the method still practised by the Persians and the Hindus, who likewise, for the sake of greater distinctness, are accustomed to separate the different ranks of figures by perpendicular lines, as in the form here annexed. Of 13,

| | | | | |
|---|---|---|---|---|
| | 3 | 7 | 0 | 9 |
| | 8 | 5 | 4 | 0 |
| | 2 | 6 | 1 | 8 |
| | | 7 | 0 | 6 |
| | 3 | 5 | 5 | 3 |
| 1 | 2 | 0 | 2 | |
| [illegible] | 5 | 5 | 7 | 3 |

the sum of the digits in the first left hand column, 3 is placed at the bottom, and 1 advanced to a higher cell in the line below; in the next column, 5 is set down, and 2 placed a step forward in the lower line; and the same process is carried through all the rest. But it is the fashion of the East, to preserve only the result of the summation, the other auxiliary rows of figures being always rubbed out. For this reason, they term the lower horizontal line, which in this example is 1202, the *Khuti Mahi*, or *obliterating line*. To such a practice, the usual mode of operating in those countries is indeed well suited. The Hindus at present, as Dr John Taylor of Bombay acquaints us, perform their numerical computations on a board about a foot long and eight inches broad. A white ground being formed by a sort of pipe-clay, the board is then covered with fine sand, or with *gulal*, that is, flour dyed of a purple colour. The figures or letters are traced by a wooden style, which, displacing the sand or purpled flour, leaves the white ground exposed. By passing the finger gently over the surface of the powder, those forms are easily effaced, and the board is again fitted for receiving new impressions. It is customary for them to rub out each successive step, even of a short process, so as to leave on the board only the general result of the operation.

The process of Addition from right to left, which is on the whole better adapted to our habits, would be rather shortened, by writing under each column the units of the sum, and below it the tens in a smaller character, which are to be joined to the figures of the next column in adding them. By a little practice, however, this precaution is rendered unnecessary, and the small subscribed figures are retained mentally, and *carried* to the successive higher columns. Above the several units, tens, hundreds, &c. are likewise placed here the marks contrived by Stevinus, to ascertain the respective ranks of the digits.

| | ③ | ② | ① | ⓪ |
|---|---|---|---|---|
| | 3 | 7 | 0 | 9 |
| | 8 | 5 | 4 | 0 |
| | 2 | 6 | 1 | 8 |
| | | 7 | 0 | 6 |
| 1 | 5 | 5 | 7 | 3 |
| 1 | 2 | 0 | 2 | |

This operation is somewhat easier, if performed with deficient figures. Thus, the numbers may be changed into others with the defects interspersed. In this mode, there being a sort of counterbalance, it will seldom be required to carry any thing to the higher columns.

| | | ③ | ② | ① | ⓪ |
|---|---|---|---|---|---|
| | | 4 | $\bar{3}$ | 1 | $\bar{1}$ |
| | 1 | $\bar{2}$ | 5 | 4 | 0 |
| | | 3 | $\bar{4}$ | 2 | $\bar{2}$ |
| | | 1 | $\bar{3}$ | 1 | $\bar{4}$ |
| | 1 | 6 | $\bar{5}$ | 8 | $\bar{7}$ |
| or | 1 | 5 | 5 | 7 | 3 |

Suppose those numbers were all transferred to the *Duodenary Scale*, they will stand thus: The summation being carried on both from right to left, and from left to right. In the latter mode, the first column gives *twenty-one*, that is, 1 dozen and 9. The rest of the operation proceeds in the same way.

| | |
|---|---|
| 2191 | 2191 |
| 4038 | 4038 |
| 1622 | 1622 |
| 40σ | 40σ |
| 7 | 19 |
| 1σ | 20 |
| 20 | 1σ |
| 19 | 7 |
| 9019 | 9019 |

If the condensed process be followed, the form will be a little different, and more distinct. The summation is also shortened somewhat, by introducing deficient figures; nor is there, in that case, any occasion for carrying, since the accumulated excess of the digits is partly counterbalanced by the intermixed defects.

| | |
|---|---|
| 2191 | 22$\bar{3}$1 |
| 4048 | 5$\bar{1}$4$\bar{4}$ |
| 1622 | 2$\bar{6}$22 |
| 40ʊ | 5$\bar{1}$$\bar{2}$ |
| 9019 | 902$\bar{5}$ |
| | or 9019 |

To facilitate the working on the *Duodenary Scale*, it would be expedient to construct an *Addition Table.* By means of this, we may at once sum up a row of pence, and carry the excess to the place of shillings.

*Duodenary Scale.*

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ʊ | ɵ | |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ʊ | ɵ | 10 | 1 |
| | 4 | 5 | 6 | 7 | 8 | 9 | ʊ | ɵ | 10 | 11 | 2 |
| | | 6 | 7 | 8 | 9 | ʊ | ɵ | 10 | 11 | 12 | 3 |
| | | | 8 | 9 | ʊ | ɵ | 10 | 11 | 12 | 13 | 4 |
| | | | | ʊ | ɵ | 10 | 11 | 12 | 13 | 14 | 5 |
| | | | | | 10 | 11 | 12 | 13 | 14 | 15 | 6 |
| | | | | | | 12 | 13 | 14 | 15 | 16 | 7 |
| | | | | | | | 14 | 15 | 16 | 17 | 8 |
| | | | | | | | | 16 | 17 | 18 | 9 |
| | | | | | | | | | 18 | 19 | ʊ |
| | | | | | | | | | | 1ʊ | ɵ |

It may serve as a check on the accuracy of Addition in the *Denary Scale*, to cast out the *nines*. Thus, in the example here adopted, the first number 3709, adding the digits 3 and 7, and passing over the 0 and 9, gives an excess of 1; the next number 8540, treated in that way, leaves 8; and the second and third numbers give respectively 8 and 4. But these excesses again, or 1, 8, 8, 4, on casting out *nine*, leave a final surplus of 3; which is the same as what arises from the separation of *nine* in the sum 15573, a presumption, though not an absolute proof indeed, of the correctness of the operation.

In the *Duodenary Scale*, the corresponding trial is made by casting out *eleven*. The several excesses are

2, 4, 0, and 2, making 8; the very same as results from the analysis of the sum 9019.

The Hindus are still totally unacquainted with these curious properties. They prove addition, by cutting off the uppermost line, and afterwards joining it to the amount of the rest, as frequently practised in Europe. But a better mode of checking any error, and correcting wrong associations of numbers, is to begin at the top, and to sum the rows of digits downwards.

It would greatly facilitate commercial transactions, if all subdivisions were carried downwards on the same scale, the decimal system being the best adapted to the prevailing mode of numeration. But as this obvious improvement is not likely to be soon embraced, an example or two shall be likewise given of the addition of mixed fractions. First in Apothecaries' Weights, the pound consisting of 12 ounces, the ounce of 8 drams, the dram of 3 scruples, and the scruple of 20 grains. Here the first column, amounting to 38 grains, leaves 18, and sends 1 to the scruples, which now make 5, or 2 scruples and 1 dram. In the same way, the rest of the process goes forward.

| ℔ | ℥ | ʒ | ℈ | gr. |
|---|---|---|---|---|
| 23 | 5 | 6 | 1 | 14 |
| 58 | 11 | 3 | 2 | 9 |
| 34 | 0 | 7 | 0 | 15 |
| 16 | 8 | 0 | 1 | 0 |
| 133 | 2 | 1 | 2 | 18 |

The next example may be drawn from measures of length, where 12 inches make a foot, 3 feet a yard, $5\frac{1}{2}$ yards a pole, 40 poles a furlong, and 8 furlongs a mile. The only difficulty here consists in adding the yards, which, with 2 carried, amount to 10; but 10 contains $5\frac{1}{2}$ *once* and leaves $4\frac{1}{2}$, and consequently 4 is set down and 1 foot 6 inches joined to the preceding columns.

| M. | F. | P. | Y. | F. | In. |
|---|---|---|---|---|---|
| 31 | 3 | 23 | 4 | 1 | 7 |
| 17 | 2 | 36 | 3 | 2 | 11 |
| 42 | 1 | 15 | 1 | 1 | 8 |
| 26 | 0 | 11 | 0 | 2 | 3 |
| 117 | 0 | 11 | 4 | 3 | 5 |

## SUBTRACTION.

THIS operation, having for its object to find the difference between two numbers, is precisely the reverse of addition. The same auxiliary table may hence answer for both. Thus, if 7 joined to 6 makes 13, it is equally clear, that 7 taken away from 13 must leave 6. For the sake of distinction, the greater of the two numbers is called the *Minuend*, and the other the *Subtrahend.*

The method of proceeding will be most clearly perceived from the inspection of an example. Let it be required to take 428053 from 702632. The process may be conducted, as in addition, either from right to left, or from left to right. In the former way, 3 cannot be taken from 2, but the effect will evidently be the same if *ten* were added to both the minuend and the subtrahend. *Ten* may therefore be joined to the 2, while *one*, as equivalent to it, is thrown to 5, which occupies the place higher. This addition of the 10 is called *borrowing*, and the countervailing addition of 1 in the next bar is called *carrying*. Take 3 then from 2, with the junction of 10 borrowed, or 12, and there remains 9, which is set down and the 1 subscribed to the next higher bar. Again, 5 is to be taken from 3, which can be done only by annexing 10, for which 1 is subtracted on the advanced bar. Consequently 8 is written down; but no carrying is required in the next bar which leaves 6. It is now required

$$\begin{array}{r} 702632 \\ \underline{428053} \\ 384689 \\ \underline{11011} \\ 274579 \end{array}$$

to subtract 8 from 2 or 12, and therefore to set down 4 and advance 1 for the 10 borrowed. Next 2 is taken from 0 or 10, leaving 8, and 1 to be subscribed under the highest bar. Lastly 4 is taken from 7, and 3 written down.—The result would evidently be the same, if the operation had proceeded by an inverted order from left to right, as commonly practised by the Hindus. The subscribed figures 1, 1, 0, 1, 1, which had been joined to the minuend to promote the subtraction, must now be taken from the digits 3, 8, 4, 6, 8, 9, which marked the excess, in order to give the true remainder 274579.

But instead of writing down those carried figures, they may with more convenience be applied mentally. In this case they are at each step either taken away from the minuend or joined to the subtrahend before the subtraction is made.

$$\begin{array}{r} 702632 \\ \underline{428053} \\ 274579 \end{array}$$

Let the same example be worked with deficient digits. In substracting the lower number, it is only required to change the character of its digits, and then add them. The operation, therefore, needs no farther explanation.

$$\begin{array}{r} 703\bar{4}32 \\ \underline{43\bar{2}153} \\ 3\bar{3}5\bar{5}8\bar{1} \\ \text{or } 274579 \end{array}$$

Next, suppose those numbers were converted into the *Duodenary Scale*. The subtraction will be performed thus: It is only to be observed, that when there is occasion for borrowing, *twelve* is joined to the digit of the minuend, and *one* is carried or annexed to the higher digit of the subtrahend.

$$\begin{array}{r} 29\tau748 \\ \underline{187871} \\ 112\tau97 \end{array}$$

Subtraction is proved by adding the remainder to the subtrahend, which should give the minuend. The mode of casting out *nines* on the *Denary Scale*, or *elevens* on the *Duodenary*, might likewise be employed. Thus, with the former, the excesses are 2 and 4, leaving 7 for the excess of the remainder; and in the latter, the excess of the minuend is 7, the subtrahend being divisible by 11.

Since the terms of a descending scale are treated in the same way, it would, perhaps, be superfluous to take examples of the subtraction of decimal fractions.

Of mixed quantities, a single example in sexagesimals may be sufficient; the remainder of this subtraction being here the difference between the chords of 144° and 96°, and therefore equal to the chord of 24°.

| | | | |
|---|---|---|---|
| 114° | 7′ | 36″ | 24‴ |
| 89 | 10 | 38 | 33 |
| 24 | 56 | 57 | 51 |

## MULTIPLICATION.

This operation, it was observed, is nothing but a repeated addition. The object which it seeks, is to collect, for a *Product*, the *Multiplicand*, as often as there are units contained in the *Multiplier*. Such a process, however, would have proved intolerably tedious, if the principle of numerical arrangement had not come to lend its aid. Suppose it were required to *double* the number 748, this would be performed by adding it *twice*; and in the same way it might be *tripled*, as here exhibited. But if the multiplier be a composite number, the operation is shortened, by an intermediate procedure. Thus, if 748 were to be multiplied by 6, it may be done, either by adding the *double* of it three times, or the *triple* of it twice: Again, the addition of this result *five* times, would give the product of the original

| | |
|---|---|
| 748 | |
| 748 | 748 |
| 748 | 748 |
| 2244 | 1496 |

| | |
|---|---|
| 1496 | |
| 1496 | 2244 |
| 1496 | 2244 |
| 4488 | 4488 |

number by 30. This product, it is obvious, could likewise be obtained by adding the *triple* of 748 ten times, and consequently by subjoining a zero. In general, since any digit in the *Denary Scale* is augmented ten fold at each move to a higher place, its product into the multiplicand must give a similar increase.

Let 748 be multiplied by the complex number 632. The multiplier now consists of 2 *units*, 3 *tens*, and 6 *hundreds ;* wherefore the compound product will be discovered, by resuming the separate products of those digits, and disposing them in their order. Here *one* zero is subjoined to the product of *tens*, *two* zeros to that of *hundreds*, and so forth. These zeros, however, may be omitted, since the value of each figure in the lower rows is determined by the position of those in the first or uppermost one. The three rows being collected together, give 472736 for the result.

$$\begin{array}{r} 1496 \\ 22440 \\ 448800 \\ \hline 472736 \end{array}$$

But the operation is abridged, by performing mentally this repetition or summation of each digit in the multiplicand. Thus, resuming the last example: 8 repeated *twice* makes 16, which is set down ; 4, repeated as often on the next advanced bar, gives 8 ; and 7 repeated likewise *twice*, and shifted a place higher, produces 14. In this way, the other rows of separate products are successively formed. The operation may be pursued either from right to left, or from left to right.

| | |
|---|---|
| 748 | 748 |
| 632 | 632 |
| 16 | 42 |
| 8 | 24 |
| 14 | 48 |
| 24 | 21 |
| 12 | 12 |
| 21 | 24 |
| 48 | 14 |
| 24 | 8 |
| 42 | 16 |
| 472736 | 472736 |

Such was nearly the procedure of the ancient Greeks. Every step in the multiplication of complex numbers, represented by alphabetic symbols, appeared as detached and separate members of the product. The operations of arithmetic, among that ingenious people, advanced like writing, from left to right, each part of the multiplier being combined in succession with the several parts of the multiplicand. The products were distinctly noted, or, for sake of compactness, grouped and conveniently dispersed, to be collected afterwards into one general amount. Pappus of Alexandria, in his valuable *Mathematical Collections*, has preserved a set of rules which Apollonius had framed, for facilitating arithmetical operations. These are, in the cautious spirit of the ancient Geometry, branched out into no fewer than twenty-seven propositions, though all comprised in the principle stated by Archimedes, *That the product of two integers of different ranks, will occupy a rank corresponding to the sum of the component orders.* Suppose, in the last example, that 30 came to be multiplied into 700 : Take the lower corresponding characters 3 and 7, which were called radicals, the one depressed *ten* times, and the other an *hundred* times; and multiply their product 21 successively by the *ten*, and by the *hundred*, or at once by a *thousand*, and the result is 21000.

These methods of multiplying numbers would become in many cases excessively tedious and perplexed. The modern practice of consolidating the figures at once on each bar before they are written down, by *carrying*, as in the process of addition, is much simpler, and decidedly preferable. Instead of noting the 1 of the first product 16, it is joined immediately to the next product 8, and the sum 9 is written down. In the second row again, *thrice* 8 makes 24, or leaves 4, and sends 2 to the product 12. The other figures are obtained in the same way.

$$\begin{array}{r} 748 \\ \underline{632} \\ 1496 \\ 2244\phantom{0} \\ \underline{4488\phantom{00}} \\ 472736 \end{array}$$

Stevinus followed in some measure the principles of Archimedes and the rules laid down by Pappus, and marked the progression of *tens* by his series of encircled exponents. In the highest product, for instance, the 7 multiplied by the 6, and both advanced to a *second* place from that of units, and therefore by *two* steps, and again by *two* more, would give 42 for the *fourth* place, and consequently promote the 4 itself to the *fifth* place.

| | | | | | |
|---|---|---|---|---|---|
| | | | 2 | 1 | 0 |
| | | | 7 | 4 | 8 |
| | | | 6 | 3 | 2 |
| | | 1 | 4 | 9 | 6 |
| | 2 | 2 | 4 | 4 | |
| 4 | 4 | 8 | 8 | | |
| 4 | 7 | 2 | 7 | 3 | 6 |
| 5 | 4 | 3 | 2 | 1 | 0 |

This now is the ordinary form of Compound Multiplication, and it seems scarcely to admit of any material improvement. But, to shorten the repeated summation of digits, it is expedient to construct a table, which must be engraved in the memory of the arithmetician. It was anciently styled the *Pinax*, or *Mensa Pythagorica*, from the name of the Philosopher who first taught the use of it to the Greeks. By those ingenious people,

MULTIPLICATION TABLE.

*Denary Scale.*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 |
| | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 2 |
| | | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 3 |
| | | | 16 | 20 | 24 | 28 | 32 | 36 | 4 |
| | | | | 25 | 30 | 35 | 40 | 45 | 5 |
| | | | | | 36 | 42 | 48 | 54 | 6 |
| | | | | | | 49 | 56 | 63 | 7 |
| | | | | | | | 64 | 72 | 8 |
| | | | | | | | | 81 | 9 |

it was likewise called the *Logistic*, or *Calculating Abacus*. It is readily formed by repeated additions, but, though now so very common, I have annexed it here. The mechanical method of multiplying digits on the hands, which has been already explained, may serve as an useful auxiliary, in fixing the recollection of the series of products.

It may be observed, that the numbers 1, 4, 9, 16, 25, 36, 49, 64, and 81, which occupy the diagonal, are the second powers or squares of the successive digits.—From the inspection of the table, we gather that 1 is the terminating figure in the *three* products, 1, 21, and 81; that 2 terminates the *six* products 2, 12, 12, 32, 42, and 72; that 3 occurs as the terminating figure in only the *two* products 3 and 63; that 4 terminates the *four* products, 4, 14, 24, and 54; that 5 terminates likewise the *five* products, 5, 15, 25, 35, and 45; that 6 is the terminating figure in the *five* products, 6, 16, 16, 36, and 56; that 7 terminates only the *two* products 7 and 27; that 8 terminates the *five* products, 8, 18, 28, and 48; and that 9 occurs only *twice* as the terminating figure, in 9 and 9. It hence follows, that, out of *thirty-four* chances, there are *six* that any composite number should end in 2; *five* chances that it should end in 5, 6, or 8; *four* chances that it should end in 4; *three* chances that it should end in 1; *two* chances that it should end in 3 or 7; and *two* chances likewise that the terminating figure should be 9. These very different proportions in the recurrence of the several digits at the end of a number, may be remarked in the large tables of products. It likewise appears, that the bulk of the prime numbers must terminate with 9, 3, or 7, and the rest with 1.

Notwithstanding the simplicity and obvious advantage of the Multiplication Table, it yet forms no part of the elementary education of the Hindus; a singular fact, which might seem to contradict the received opinion, that Pythagoras brought the knowledge of it, with other higher acquisitions, from the East. The boys in India are, no doubt, obliged to supply the want of this important help, by the tedious process of repeated additions, till practice at last renders them familiar with the products of the ordinary digits. In like manner, our young scholars are now left to grope their way without a guide through Addition, till the experience of many trials makes them acquainted with all the binary combinations of the lower numbers.

It may be instructive, to compare the operation of an example of compound multiplication in the ordinary way, with another performed by deficient figures. In this instance, the working is evidently easier with the deficient figures, since lower digits are adopted in the multiplication. But it must be observed, that a deficient figure reverses the character of all the digits which it multiplies. The restoration of the ordinary figures is better understood, if, as here, it be made by successive steps.

$$
\begin{array}{rr}
4819 & 5\bar{2}2\bar{1} \\
\underline{378} & \underline{4\bar{2}\bar{2}} \\
38552 & \bar{1}04\bar{4}2 \\
33733\phantom{0} & \bar{1}04\bar{4}2\phantom{0} \\
\underline{14457\phantom{00}} & \underline{20\bar{8}8\bar{4}\phantom{00}} \\
1821582 & 2\bar{1}\bar{8}2\bar{4}\bar{2}2 \\
 & \text{or } 19\bar{8}16\bar{2}2 \\
 & \text{and } 1821582
\end{array}
$$

Another example, the same as what afterwards occurs, may be likewise produced. By a little practice, the working with deficient figures would evidently become easier, and more expeditious than the common way.

$$
\begin{array}{rr}
 & 4\bar{3}0\bar{2}5 \\
36985 & 1\bar{4}43\bar{2} \\
\underline{6428} & \underline{\bar{8}6030} \\
295880 & 12\bar{9}0\bar{5}5\phantom{0} \\
73970\phantom{0} & 15\bar{2}0\bar{6}0\phantom{00} \\
147940\phantom{00} & \bar{1}\bar{5}2060\phantom{000} \\
\underline{221910\phantom{000}} & \underline{4\bar{3}0\bar{2}5\phantom{0000}} \\
237739580 & 3\bar{7}8\bar{2}\bar{6}0\bar{5}80 \\
 & \text{or } 237739580
\end{array}
$$

Deficient figures applied to the multiplication of the digits themselves, furnish nearly the same practical rule as in Palpable Arithmetic. Thus, to multiply 9 by 6, the operation may be performed, by substituting for them $1\bar{1}$ and $1\bar{4}$, or 10 diminished by 1 and by 4. The result is consequently $1\bar{5}4$, or 54. But it appears obvious, that the last figure 4 is the product of the deficient *units*, $\bar{1}$ and $\bar{4}$, and that the two first figures $1\bar{5}$ or 5, denoting *tens*, are merely 10 less than the sum of $1\bar{1}$ and $1\bar{4}$, or of 9 and 6. This accordingly

$$
\begin{array}{r}
1\bar{1} \\
\underline{1\bar{4}} \\
\bar{4}4 \\
\underline{1\bar{1}\phantom{4}} \\
\underline{1\bar{5}4} \\
54
\end{array}
$$

is the rule given by the Arabian and Persian authors. It was farther simplified, however, in the earlier treatises on arithmetic published in Europe. Orontius Finæus, of Briançon, Professor of Mathematics at Paris, who wrote his little tract in 1525, directs a cross to be drawn, on the one side of which the digits 9 and 6 are placed, and opposite to them, on the other side, their defects from 10, or 1 and 4, are set down. Then 4 is multiplied into 1, and the product 4 noted under them, while, following the oblique arms of the cross, 4 is taken from 9, or 1 from 6, to leave 5 for the place of tens. But it is evident, that to subtract 4, or the excess of 10 above 6, from 9, is precisely the same thing as to add 6 and 9 together, and then take away 10, as in the previous rule. The other variation of the process, by taking 1 from 6, must likewise give a similar result.

$$\begin{array}{cc} 9 & 1 \\ 6 & 4 \\ \hline 5 & 4 \end{array}$$

As another example, a little more difficult, suppose 7 were to be multiplied by 6. These digits are equivalent to 13 and 14, so that their product is 162 or 42. With the cross, the 4 multiplied into 3 gives 12, which leaves 2 units, and advances 1 to the rank of tens; and 6 diminished by 3, or 7 diminished by 4, supply 3 additional tens, making up 42 as before.

$$\begin{array}{r} 13 \\ 14 \\ \hline 12 \\ 4 \\ 13 \\ \hline 162 \\ \hline 42 \end{array} \qquad \begin{array}{cc} 7 & 3 \\ 6 & 4 \\ \hline 1 & 2 \\ 3 & \\ \hline 4 & 2 \end{array}$$

The cross might also be accommodated to the multiplication of numbers exceeding ten. Thus, the product of 13 by 12 is found by setting opposite to them 3 and 2, their *excesses* above 10; then multiplying the 3 by 2 for units, and *adding* cross-wise the 2 to 13, or the 3 to 12

$$\begin{array}{cc} 13 & 3 \\ 12 & 2 \\ \hline 15 & 6 \end{array}$$

for tens. The latter part of the operation is evidently the same thing, as adding the numbers together, and deducting ten from their amount; a rule which the Persians also employ in this case.

Let this method of multiplication be applied to the squares of the digits from 9 to 6 inclusive. The operation is performed with great facility, the deficient figure being squared for units, and subtracted from its corresponding digit to express the tens.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 9 | 1 | 8 | 2 | 7 | 3 | 6 | 4 |
| 9 | 1 | 8 | 2 | 7 | 3 | 6 | 4 |
| 8 | 1 | 6 | 4 | 4 | 9 | 3 | 6 |

It hence appears, that a square must have the same terminating figure, if the root end in 1 or 9, in 2 or 8, in 3 or 7, in 4 or 6. It likewise follows, that all square numbers terminate in these five digits, 1, 4, 5, 6, 9, which lie equally distant on each side of the middle one 5, from which they differ by 1 and 4. When a number ends in 1, its square root must end in 1 or 9; when it ends in 4, the root ends in 2 or 8; when it ends in 5, the root will also end in 5; when it ends in 6, the root will end in 4 or 6; and when it ends in 9, the root will end in 3 or 7. Unless, therefore, in the case of 5, there are always two corresponding terminations of the root, making together the number *ten*.

The Arabian and Persian, and likewise the earlier European writers on Arithmetic, enumerate several different ways of performing Multiplication. Of these, it may be proper to select the most remarkable varieties.

1. The rudest mode was that of *Cross-Multiplication*, in which the distinct products of all the digits of the multiplier and of the multiplicand in every direction, are se-

parately set down. No figures are *carried* in this process, which is yet very complex and embarrassed. It was the method also that the Greeks were obliged to follow, for want of a simple denary notation, and has been already illustrated in the first example.

2. The commonest way was perhaps the *Diced or Tesselated Multiplication.* This differs nothing indeed from the ordinary method, except that the digits of the products, as they successively arise, are deposited in cells. The only advantage of such a form of operation, consists in the preserving better the regular alineation of the figures.

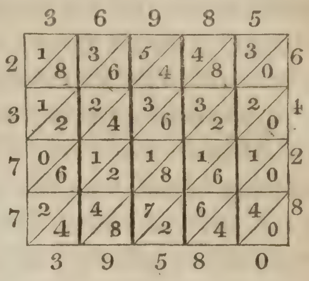
| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 3 | 6 | 9 | 8 | 5 |
| | | | | | 6 | 4 | 2 | 8 |
| | | | 2 | 9 | 5 | 8 | 8 | 0 |
| | | | 7 | 3 | 9 | 7 | 0 | |
| | 1 | 4 | 7 | 9 | 4 | 0 | | |
| 2 | 2 | 1 | 9 | 1 | 0 | | | |
| 2 | 3 | 7 | 7 | 3 | 9 | 5 | 8 | 0 |

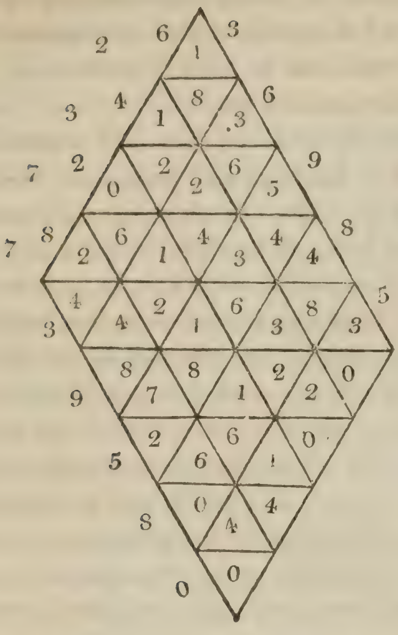
3. Another mode of *Multiplication* was called the *Quadrilateral* or *Square.* In this arrangement, it was not requisite to advance the products according to the place of the corresponding figure in the multiplier. The same effect was produced by an oblique summation. Beginning on the right hand at the top, the 0 is marked at the corner of its cell; the 8 and 0 give 8, for a place lower; 8, 7, 0, throw 5 to the step below this, with 1 to be carried to the next addition, in the same slanting direction.

| | | | | | | |
|---|---|---|---|---|---|---|
| | 3 | 6 | 9 | 8 | 5 | |
| | | 6 | 4 | 2 | 8 | |
| 2 | 9 | 5 | 8 | 8 | 0 | |
| | 7 | 3 | 9 | 7 | 0 | 0 |
| 1 | 4 | 7 | 9 | 4 | 0 | 8 |
| 2 | 2 | 1 | 9 | 1 | 0 | 5 |
| 2 | 3 | 7 | 7 | 3 | 9 | |

4. The last form of *Multiplication* which deserves any particular notice, is what was styled by Lucas de Burgo, the *Latticed*, or *Rhomboidal.* In this operation, the products of all the digits are severally dispersed in lozenges, or in square cells divided by diagonal lines, a form of procedure

by which the fatigue of *carrying* to the higher places is entirely spared. The method, however, admits of some variation. 1. The multiplier and multiplicand may be written along the top, and down the left hand side of an oblong, which is subdivided into square cells, these again being parted by diagonals running obliquely downwards from right to left. The multiplication begins at the left corner above, and the successive products are inscribed in the cellular triangles of each horizontal zone. The summation is then performed along the diagonal lines. This figurate process was followed by the Hindus and Persians, among whom it obtained the technical name of *Shabakh*. 2. Another variation of the general mode consisted in writing the multiplier along the top, and the multiplicand down the left hand of a divided quadrangle, the products beginning with the units, and proceeding along the horizontal columns from right to left; the summation then sets out from the right corner, and runs up slanting to the left.—This mode of operating was peculiar to the Arabians and Persians, and by them communicated to the Hindus, who occasionally use it. On the next page is an example borrowed from the judicious travels of Sir John Chardin. Suppose it were sought to multiply the number 36985 by 6428. The Persian Arithmeticians, having drawn a rhomboid, would, beginning at the top, write these numbers downwards along the upper sides, and then divide the figure into equilateral triangles, by combining oblique with horizontal lines.





```
    36985
     6428
---------
18
 36
  54
   48
    40
 12
  24
   36
    32
  6
  12
   18
    16
     10
  24
   48
    72
     64
      40
---------
237739580
```

Now, the multiplication is carried along the rows on the left side of the rhomboid: 6 into 3 gives 18, which is disposed in the uppermost triangle and the one below it; 6 into 6 gives 36, which is deposited in the two next triangles; and the same process is continued through the series. Again, 4 times 3 makes 12, which is placed in the two uppermost triangles of the next row. The rest of the operation of filling the triangles is easily understood. But to collect the products, the figures in each horizontal row, beginning at the bottom, are added up, and the tens carried to the one immediately above it. Thus, the zero at the point of the rhomboid remains unchanged; in the row above this, 4, 4, 0 make 8; in the next row, 2, 6, 6, 1, 0 make 15, and 5 being set down, the 1 is carried to the higher row, 8, 7, 8, 1, 2, 2, 0, making 29, of which 9 is set down, and 2 carried to the row above it. In this way, the summation is quickly performed, giving 237739580 for the complete product.

It will be admitted, however, that such artificial helps may prove useful in laborious and protracted multiplications, by sparing the exercise of memory, and preventing the attention from being overstrained. Of this description are the *Rods* or *Bones*, which we owe to the early studies of the great Napier, whose life, devoted to the improvement of the science of calculation, was crowned by the invention of logarithms, the noblest conquest ever achieved by man. These rods were small squared pieces of ivory or bone, box or silver, about three inches long, and only three-tenths of an inch in breadth and thickness. On their four sides, were engraved the successive columns of the common multiplication table, the cells being parted by diagonal lines running obliquely downward from right to left.—But instead of rods presenting different surfaces,

| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | /2 | /4 | /6 | /8 | 1/0 | 1/2 | 1/4 | 1/6 | 1/8 | /0 |
| 3 | /3 | /6 | /9 | 1/2 | 1/5 | 1/8 | 2/1 | 2/4 | 2/7 | /0 |
| 4 | /4 | /8 | 1/2 | 1/6 | 2/0 | 2/4 | 2/8 | 3/2 | 3/6 | /0 |
| 5 | /5 | 1/0 | 1/5 | 2/0 | 2/5 | 3/0 | 3/5 | 4/0 | 4/5 | /0 |
| 6 | /6 | 1/2 | 1/8 | 2/4 | 3/0 | 3/6 | 4/2 | 4/8 | 5/4 | /0 |
| 7 | /7 | 1/4 | 2/1 | 2/8 | 3/5 | 4/2 | 4/9 | 5/6 | 6/3 | /0 |
| 8 | /8 | 1/6 | 2/4 | 3/2 | 4/0 | 4/8 | 5/6 | 6/4 | 7/2 | /0 |
| 9 | /9 | 1/8 | 2/7 | 3/6 | 4/5 | 5/4 | 6/3 | 7/2 | 8/1 | /0 |

mere slips of ivory could likewise be used, the one of them containing the row of digits being termed the *index rod.*

Let us resume the former example. The rods, inscribed with the several figures of the multiplicand, 36985, are selected and set in the same order, with the index placed before them; then, opposite to the several figures of the multiplier, 6428, on the index, but going backwards, the numbers in each horizontal column are taken, the pair of digits in each rhombus, or double triangle, being always added; and finally, these rows of the products corresponding to each digit of the multiplier, being transcribed and properly disposed, are collected into one sum. Thus, opposite to 8, the last digit of the multiplier, and proceeding from the right along the horizontal column, there occur these figures: first 0; then 4 and 4, or 8; 6 and 2, or 8; 7 and 8, or 15; and 1 carried to 4 and 4, make 9; and lastly, the 2. The other rows are easily formed in the same way.—It is obvious, that if the horizontal columns op-

| 1 | 3 | 6 | 9 | 8 | 5 |
|---|---|---|---|---|---|
| 2 | /6 | 1/2 | 1/8 | 1/6 | 1/0 |
| 3 | /9 | 1/8 | 2/7 | 2/4 | 1/5 |
| 4 | 1/2 | 2/4 | 3/6 | 3/2 | 2/0 |
| 5 | 1/5 | 3/0 | 4/5 | 4/0 | 2/5 |
| 6 | 1/8 | 3/6 | 5/4 | 4/8 | 3/0 |
| 7 | 2/1 | 4/2 | 6/3 | 5/6 | 3/5 |
| 8 | 2/4 | 4/8 | 7/2 | 6/4 | 4/0 |
| 9 | 2/7 | 5/4 | 8/1 | 7/2 | 4/5 |

| | |
|---|---|
| 295880 | 8 |
| 73970 | 2 |
| 147940 | 4 |
| 221910 | 6 |
| 237739580 | |

posite to 8, 2, 4 and 6, were supposed to be detached and combined into an oblique group, the similarity to the Persian mode would be very striking.

But, without formally adopting either the figurate rods or the rhomboidal cells, it will sometimes be convenient, in very long multiplications, to form, by successive additions, an extemporaneous tablet of the digital products of the multiplicand. The application of this help is easily conceived.

A very neat method of trying the accuracy of any process of multiplication, consists in casting out the *nines*.—Since any number must always be composed of repeated *nines* with some remainder, every multiplier and multiplicand are only certain multiples of *nine*, with corresponding excesses. Wherefore, their compound product will contain some involved multiples of *nine*, with the product of those excesses. Or, conceive the numbers to be multiplied, were exhibited on the *Nonary Scale*; the first bar, or that of units, would evidently receive the product of the remainders of their division by *nine*. This reasoning is quite general, and must consequently apply likewise to the casting out of *eleven* or of *seven*. But the remainder of the division of any number by *nine* or *eleven* is readily found in the way before explained; and it was likewise shown how the *sevens* could be cast out.

To illustrate the application of the principle, let the first example be resumed. It was found, that 748 multiplied by 632, give 472736. But the *nines*, being cast out of the multiplier and multiplicand, leave 1 and 2, and out of the product, 2; which affords a strong presumption, though not an absolute proof, that the operation had been correctly performed. In the next example, the multiplier 378 is divisible by 9, and so is likewise the product 1821582. In the third example, 9 being cast out of 36985 and 6428, leaves 4 and 2; but on casting 9 also out of the product 237739580, the remainder is 8, or the product of 4 and 2, as it ought to be.

Again, the multiplicand 748 in the first example being divisible by *eleven*, the product 472736 is likewise divisible, for the alternate figures 6, 7, and 7 make 20, and the other alternate figures 3, 2, and 4, give 9 or 11 less. In the second example, if *eleven* be cast out of 4819, there will remain 1, and out of 378 there will remain 4; but the product 1821582 likewise leaves 4 on being divided by eleven. In the last example, *eleven* being cast out of 36985 and 6428, gives the remainders 3 and 4, which being multiplied, make 12 or an excess of 1: But *eleven*, on being thrown out of the product 237739580, leaves also 1. In like manner, if *seven* be cast out of 748 and 632, the excesses are 6 and 2, which give 12 or 5, for the remainder of the division of the product 472736 by 7. In the next example, the multiplier 378 is divisible by 7, and so is the product 1821582. Lastly, if *seven* be cast out of 36985 and 6428, the excesses are 4 and 2, which give 8 or 1 for the remainder of the division of 237739580 by 7.

The last of these methods of proving multiplication is sometimes rather tedious, though greatly simplified by attending to the order of the recurring series, 1, 3, 2, 6, 4, 5. The casting out of the *elevens* may be conjoined with that of casting out the *nines*, to strengthen the assurance of the accuracy of an operation. But the latter mode is the one most generally adopted. This elegant numerical property was known to the Arabian writers on arithmetic, who styled it the *Tarazu or Balance.* Yet the Hindus are still unacquainted with it, and have no other way of proving multiplication but by reversing the process itself, and converting it into division.

It is evident, from the nature of notation, that, in the descending scale, the products corresponding to each figure of the multiplier, instead of being advanced, should be shifted backwards. Hence the common rule for the multiplication of decimal fractions—to cut off as many decimals as are found in both factors. But, since the remote decimals are of trifling import, a very commodious abbreviation is, to begin the process at the place of units, and reject the very low terms. Passing to the 6 of the multiplier, the last figure 6 of the multiplicand is struck off; but, as it would have given 36, the nearest whole number 4, expressing the tens, is carried to the product of 6 into 3, making 22. The same thing is repeated at each multiplication.

```
4.236
1.618
-----
4.236
2.542
   42
   34
-----
6.854
```

In farther illustration of the process of multiplication, we shall perform the same examples on

MULTIPLICATION TABLE.

*Duodenary Scale.*

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ϭ | ɵ | 1 |
| | 4 | 6 | 8 | ϭ | 10 | 12 | 14 | 16 | 18 | 1ϭ | 2 |
| | | 9 | 10 | 13 | 16 | 19 | 20 | 23 | 26 | 29 | 3 |
| | | | 14 | 18 | 20 | 24 | 28 | 30 | 34 | 38 | 4 |
| | | | | 21 | 26 | 2ɵ | 34 | 39 | 42 | 47 | 5 |
| | | | | | 30 | 36 | 40 | 46 | 50 | 56 | 6 |
| | | | | | | 41 | 48 | 53 | 5ϭ | 65 | 7 |
| | | | | | | | 54 | 60 | 68 | 74 | 8 |
| | | | | | | | | 69 | 76 | 83 | 9 |
| | | | | | | | | | 84 | 92 | ϭ |
| | | | | | | | | | | ϭ1 | ɵ |

the *Duodenary Scale*. It will be convenient, however, though not quite essential, to construct previously a table of products.

Let it be required then to multiply the number 4819 by 378. Transformed into the *Duodenary Scale*, they will stand thus: The same operation is likewise here performed by deficient figures.

| | |
|---|---|
| 2957 | 3365 |
| 276 | 356 |
| 14896 | 15406 |
| 17631 | 12141 |
| 5602 | 9853 |
| 73ʊ1ʊ6 | 89ʊ106 |
| | 73ʊ1ʊ6 |

Suppose the large numbers 36985 and 6428 were now converted into the *Duodenary Scale*, their multiplication, in common and in deficient figures, would proceed in this manner.

| | |
|---|---|
| 194ʊ1 | 23521 |
| 3878 | 4484 |
| 123288 | 71884 |
| 1059ʊ7 | 139348 |
| 123288 | 71884 |
| 54263 | 71884 |
| 67750938 | 75871344 |
| | 67750938 |

The mode of proof which in the *Denary Scale* employed *nine* and *eleven*, will evidently be *eleven* and *thirteen* on this *Denary Scale*. Thus, in the former example, *eleven* being cast out of 2957 leaves 1; and out of 276 it leaves 4; but the product 73ʊ1ʊ6 divided by *eleven*, gives likewise a remainder of 4. If *thirteen* be thrown out of those factors, it leaves 9 and 1; and out of the product, it also leaves 9. In the latter example, *eleven* divides 194ʊ1 and 3878, with remainders of 3 and 4, but it likewise divides 67750938, leaving 1, the same in effect as *twelve*, the product of 3 and 4. Suppose *thirteen* to be cast out of the factors, the remainders are 9 and 6, which being multiplied, give *fifty-four*, or an excess of 2; but the product 73ʊ1ʊ6 likewise gives the same excess.

But the *Duodenary Scale* is of greater consequence, when viewed in the descending progression, since the subdivision by twelves has, to a certain extent, obtained currency in the denomination of money, and of weights and measures. A few examples will explain the management of these fractions. Suppose it were sought to multiply L. 6 : 15 : $3\frac{3}{4}$, by 53, the operation with Duodecimals will be performed thus:

$$\begin{array}{r} 6.923 \\ 45.\phantom{000} \\ \hline 29.9ɸ3 \\ 230.90\phantom{0} \\ \hline 25ʊ.6ɸ3 \end{array}$$

The product 25ʊ.6ɸ3, converted again into the *Denary Scale*, gives L. 358 : 11 : $6\frac{1}{2}$. This result is more easily brought out than by Decimals, as in the operation here annexed.—If the Pound Sterling had been divided into 12 shillings, as the shilling is into 12 pence, the application of Duodecimals to accounts would have been extremely convenient.

$$\begin{array}{r} 6.765625 \\ 53.\phantom{000000} \\ \hline 20.296875 \\ 338.28125\phantom{0} \\ \hline 358.578125 \end{array}$$

Duodecimals are best adapted, however, to practical mensuration, where feet, inches and their subordinate parts, are brought into play. Thus, if it were required to find the solid contents of a log of timber, 2 feet $7\frac{1}{4}$ inches square, and 27 feet $5\frac{1}{2}$ inches long. The successive multiplications would be performed in this way, all the figures below the fourth place being excluded, as of little significance. The result is 136 cubic feet, with about $2\frac{1}{2}$ twelfth parts, most inaccurately called, in this case, cubic inches.

$$\begin{array}{r} 2.73\phantom{00} \\ 2.73\phantom{00} \\ \hline 5.26\phantom{00} \\ 1.629\phantom{0} \\ 799 \\ \hline 6.9469 \\ 23.56\phantom{00} \\ \hline 116.9160 \\ 18.4183 \\ 2.9ʊʊʊ \\ .3483 \\ \hline 136.2694 \end{array}$$

But these fractions will likewise readily apply to the mensuration of round timber; for the relation of the circle to its circumscribing square would be expressed in duodecimals by the number .9512. Suppose, for example, a cylinder 4 feet $2\frac{3}{4}$ in diameter, and 41 feet $10\frac{1}{2}$ long.

The operation is thus performed: Four places of duodecimals only are retained, though in actual practice two places may generally be reckoned more than sufficient. The area of the circular base hence exceeds by a minute fraction 12 square feet, but the solid contents of the cylinder amounts extremely nearly to 40́4$\frac{2}{3}$, or about seven hundred cubic feet.—A similar mode of proceeding could easily be extended to the mensuration of cones and of spheres.

| | |
|---|---|
| 4.29 | |
| 4.29 | |
| 14.00 | 12.0723 |
| .856 | 41.0́6 |
| .3209 | 482.4900 |
| 15.0́769 | 12.0723 |
| .9512 | 0.8500 |
| 11.4081 | .6037 |
| .7552 | 40́4.70́50́ |
| 160 | |
| 30 | |
| 12.060́3 | |

To facilitate the operations with sexagesimals, it seemed indispensable to have a more extensive multiplication table, that should include the mutual products of all the numbers from one to sixty. Such a table was actually constructed early in the seventeenth century by Philip Lansberg, a Dutch clergyman, who resided at Middleburg. It was nearly about the same time printed likewise in the *Arithmetic* of Adrian Metius, and has been since exhibited under various forms by Dr Wallis and others. The table of sexagesimal products, however, is now comparatively of small utility, since the practice of those fractions has at last fallen into almost total disuse. It might be conveniently superseded by a table of products, carried as far as *one hundred*, which, if rightly managed, would vastly abridge and expedite the most laborious arithmetical operations. The daily habit of using such a table could not fail to imprint a great part of it on the memory of the diligent scholar,—an acquisition of immense value, both in the pursuits of science and of commerce. To every person, it would give facility and correctness in their calculations. In this view, therefore, I have, with very considerable trouble and expense, framed an extended table of products to accompany this volume. Its construction will be easily understood, and to render it more compact, I have omitted the rows of the products from 1 to 10, which are easily supplied. Along the upper horizontal line is placed the multiplicand, and, on the vertical one towards the right hand, the multiplier; and opposite to them, the product occurs in a descending column. Suppose a former example were resumed: To multiply 36985 by 6428. These numbers are to be distinguished into periods of two figures each. Beginning then at the left hand, 28 times 85 gives 2380, which is set down; 28 times 69 gives 1932, which is also set down two figures in advance; and 28 times 3 gives 84 for the advanced place. Again, the products by 64 are 5440, 4416, and 192, which are successively advanced two places.

```
    36985
     6428
     2380
   1932
  84
    5440
  4416
192
237739580
```

Let it be required to multiply the decimal fraction .8134732 by .9135455, the former of which is double the sine, and the latter the cosine, of the arc of 24°. Here the multiplication begins at the left hand, and all the figures of the product beyond the seventh place are excluded. But 91 multiplied into 81, 34, 73, and 2, gives respectively 7371, 3094, 664 and 2, which, at each step, are moved two places backwards. Again, 35 multiplied into 81, 34, 73, gives 2835, 119, and 3, which are all drawn back two places. Then, 45 being multiplied by 81 and 34, gives 364 and 2. And, lastly, 5 into 81 gives 4. The several distinct products

```
.8 1 3 4 7 3 2
.9 1 3 5 4 5 5
.7 3 7 1
     3 0 9 4
         6 6 4
             2
     2 8 3 5
         1 1 9
             3
         3 6 4
             2
             4
.7 4 3 1 4 4 8
```

being now collected, the amount is .7431448, which consequently expresses the sine of the double arc, or 48°.

This table may likewise be accommodated to the operations of sexagesimals, for nothing is wanted but to convert the products in seconds, thirds, fourths, &c. into the numbers of a higher dimension by an easy division of 60. Thus, resuming the last example, the cosine of the arc of 24° is expressed sexagesimally by 54° 48′ 45″ 50‴, and the double of its sine is 48° 48′ 30″ 12‴; making the latter, therefore, the multiplier, because the part 48 appears repeated, the operation will proceed thus: Beginning at the left hand, the product of 48° into 54° is 2592′, or 43° 12′, which is set down; the product of 48° into 48′ is 2304″, or 38′ 24″, which is set down; the product of 48° into 45″ is 2160‴, or 36″, which is also set down; and the product of 48° into 50‴ is 2400$^{iv}$, or 40‴, which is set down after the 36″. Again, the products of 48′ into the several terms of the multiplicand, are the same as before, only removed all one place lower; consequently 37‴ is substituted for 36‴ 40$^{iv}$, as the nearest integral value. Next, 30″ multiplied into the terms of the multiplicand evidently reduces them to one half, and throws them two places lower. Lastly, 12‴ multiplied into 55°, the nearest value in whole numbers of the multiplicand, in effect divides it by five, and throws the quotient 11 three places lower. The amount of the whole is hence 44° 35′ 19″ 16″; and therefore the double of this sine, or 89° 10′ 38″ 32‴, must express the chord of

| | | | |
|---|---|---|---|
| .54° | 48′ | 45″ | 50‴ |
| .48 | 48 | 30 | 12 |
| .43 | 12 | | |
| | 38 | 24 | |
| | | 36 | 40 |
| | 43 | 12 | |
| | | 38 | 24 |
| | | | 37 |
| | | 27 | 24 |
| | | | 11 |
| .44° | 35′ | 19″ | 16‴ |

96°. Ptolemy, going no farther than seconds, makes it 89° 10′ 39″.

The Arabians performed the multiplication of sexagesimals by help of square cells, parted downwards from left to right by diagonal lines. The multiplicand being placed along the top of the quadrangle, the multiplier ascended on the right side, and the operation of multiplying them proceeded from right to left, as customary in the writing of those Orientals. In short, this process was exactly the reverse of the ordinary mode followed in the multiplication of numbers on the *Denary Scale*, which they had adopted, probably without any change or modification, from the Hindus.

## DIVISION.

THIS process, being merely the reverse of Multiplication, consists in *subtracting* one number repeatedly from another. The former is called the *Divisor*, the latter the *Dividend*, while the answer, signifying how often the subtraction needs to be made, is termed the *Quotient*. The principle of numerical arrangement suggests the means of abridging this operation. Suppose it were sought to divide 1554 by 37: Let 37 be subtracted in succession from 155, which, standing one place higher than the units, corresponds to tens; the several subtractions are marked by I, II, III, and IV, which belong to the place of tens, and from remainder 7 with 4

| | | | |
|---|---|---|---|
| | 1554 | | 74 |
| I. | 37 | I. | 37 |
| | 118 | | 37 |
| II. | 37 | II. | 37 |
| | 81 | | 0 |
| III. | 37 | | |
| | 44 | | |
| IV. | 37 | | |
| | 7 | | |

annexed to it, the divisor 37 is again subtracted twice:—Whence the quotient is 42, or the number of times that 37 is contained in 1554, or must be subtracted before it exhausts this dividend.

But such an operation is evidently circuitous. The most obvious improvement is to frame, as in compound multiplication, a small tablet of the digital products of the divisor, and to subtract always the nearest less number from the successive terms of the dividend and the remainder. Let it be required to divide 22028148 by 423. The tablet of products is formed by the successive addition of the divisor 423 and its multiples; of these, the number opposite to 5 comes nearest to the first four terms of the dividend; and the remainder 87, with the next figure annexed to it, is approached the nearest by 846, the next remainder 32 with the annexed 1 is less than the divisor, and therefore, a zero is put in the quotient to preserve the place, and the following figure 4 is joined. The rest of the operation is easily conceived.

| | | |
|---|---|---|
| 1 | 423 | )22028148(52076 |
| 2 | 846 | 2115 |
| 3 | 1269 | 878 |
| 4 | 1692 | 846 |
| 5 | 2115 | 3214 |
| 6 | 2538 | 2961 |
| 7 | 2961 | 2538 |
| 8 | 3384 | 2538 |
| 9 | 3807 | 0 |

This method, however, is more tedious than needful, unless the quotient should consist of several figures. In other cases, a little practice will show how to choose the proper multiples of the divisor. The dots placed under the figures of the dividend as fast as they are taken down, or annexed for a new division, point out the ranks of the divisor. Deficient figures may likewise be sometimes introduced with advantage. An example will explain this: Suppose 1797848 were to be divided by 472. With

$$
\begin{array}{rl}
472)1797848(3809 & \\
1416\cdots & \\
\hline
3818 & \\
3776 & \\
\hline
4248 & \\
4248 & \\
\hline
0 &
\end{array}
\qquad
\begin{array}{r}
5\bar{3}2)180\bar{2}\bar{2}5(4\bar{2}1\bar{1} \\
19\bar{2}8\cdots \\
\hline
\bar{1}10\bar{2} \\
\bar{1}06\bar{4} \\
\hline
425 \\
5\bar{3}2 \\
\hline
53\bar{2} \\
53\bar{2} \\
\hline
0
\end{array}
$$

the deficient figures, the operation is somewhat easier, tho' it was here unnecessary to make any alteration on the dividend itself. In the course of working, it will often happen that the product of the divisor, after being written down, will appear greater, instead of less, than the part of the dividend from which this is to be taken; but without substituting a lower product, the oversight would be rectified by a deficient digit at the next step. Thus, instead of the first two figures 38 of the quotient, we obtain $4\bar{2}$, which is equivalent.

The Arabians and Persians perform division like multiplication by a figurate process, in which every step is distinctly set down. A sufficient number of equidistant vertical lines being drawn, another horizontal line near the top of the board is made to intersect them, and, immediately under it, is placed the dividend, the divisor being set down at such a distance below as may allow space for the operation being repeated at each step on a lower bar. Having found how often the first figure of the divisor is contained in the corresponding part of the dividend, the quotient is placed above the horizontal line, opposite to the termination of the divisor, and now multiplied into each of those digits in succession, and the products subtracted from the dividend. The divisor is then shifted upwards a step farther back, and the process recommences again.

An example will show this complex mode of proceeding. Suppose, as before, that 1797848 were to be divided by 472. Eight vertical bars being drawn, the figures of the dividend are inserted across the top, and those of the divisor at the bottom, the 4 being set opposite to 17, which it divides. The quotient 3 is then placed in the same column with 2, the termination of the divisor, and multiplied into each figure in succession. The products 12, 21, and 6 are separately subtracted, leaving 381, &c. for a new dividend. The divisor 472 is now repeated a step backwards, and the operation renewed, the next digit of the quotient being 8, and its successive products 32, 56, and 16.—This method of performing division, though unnecessarily tedious, requires no effort of memory. It is also sometimes a little varied.

| | | | 3 | 8 | 0 | 9 |
|---|---|---|---|---|---|---|
| 1 | 7 | 9 | 7 | 8 | 4 | 8 |
| 1 | 2 | | | | | |
| | 5 | 9 | | | | |
| | 2 | 1 | | | | |
| | 3 | 8 | 7 | | | |
| | | | 6 | | | |
| | 3 | 8 | 1 | | | |
| | 3 | 2 | | | | |
| | | 6 | 1 | | | |
| | | 5 | 6 | | | |
| | | | 5 | 8 | | |
| | | | 1 | 6 | | |
| | | | 4 | 2 | | |
| | | | 3 | 6 | | |
| | | | | 6 | 4 | |
| | | | | 6 | 3 | |
| | | | | | 1 | 8 |
| | | | | | 1 | 8 |
| | | | | 4 | 7 | 2 |
| | | | 4 | 7 | 2 | |
| | | 4 | 7 | 2 | | |
| | 4 | 7 | 2 | | | |

The mode of compound division, as now practised among the Hindus, appears still more involved and laborious, only the figures of the dividend and its remainders are obliterated as fast as the operation proceeds. Resuming the former example, the divisor 472 is placed under a corresponding portion of the dividend 1797848, in which it is contained 3 times; then thrice 2 is 6, which, taken from 7, leaves 1, to be written above it; thrice 7 is 21, which, taken from 79, leaves 58 above it; and thrice 4 is 12, which, taken from the 15, leaves 3. The remainder for the next division is, therefore, 3818, which contains the divisor, repeated one place farther back, 8 times.

```
      3
      4
    320
    58160
  1797848(3809
    472230
     472
      472
       472
```

Now, 8 times 2 is 16, which, taken from 18, leaves 0 and 2 to be placed above and below those figures; again, 8 times 7 is 56, which, taken from 80, leaves 24; and 8 times 4 is 32, which extinguishes that number. The remainder for a new division is only 4248, which, passing over one place, contains the next divisor, when shifted to the end, 9 times; but 9 times 2 is 18, which, taken from 48, leaves 30; 9 times 7 is 63, which cancels the 3, and takes 6 from the 42, leaving only 36; and lastly, 9 times 4 is 36, which is therefore extinguished on the board. All the figures being successively obliterated, the result only of the operation is retained.

Almost the same crowded and intricate mode of performing division had early prevailed in Europe, and even maintained its ground till about the beginning of the last century, since Dr Wallis constantly used it. This form of proceeding was by the Italians, according to Lucas de Burgo, styled the *galley*, either from the swiftness of its operation, as he thinks, or rather from the tapering shape which the group of digits acquires in the course of the work. In farther illustration of the process, I shall select an example from the numerous calculations by Regimontanus, in his tract relative to the quadrature of the circle, written as early as 1464, but not published until 1532. The question here proposed is to divide 18190735 by 415; the divisor being repeated at every step backwards, and standing under the dividend, and all the figures erased in succession. The quotient 43833 is placed down the side, and along the bottom, the remainder 50 being the only digits left on the board.

```
      11
     3134
   154750 |4
   276548 |3
  18190735|8
   4155555|3
    41111 |3
     444  |
 ---------
 43833
```

In the preceding examples the process of division is complete; but should there be any remainder, it is evident that, by annexing successive ciphers, the operation might still be carried on along the descending scale. Hence the process for converting vulgar into decimal fractions. A few examples will elucidate this practice.

```
(1.) 64)15.000000(.234375        (2.) 625)23.0000(.0368
        128·····                          1875··
        ---                               ----
        220                               4250
        192                               3750
        ---                               ----
         280                               5000
         256                               5000
         ---                               ----
         240                                  0
         192
         ---
          480
          448
          ---
           320
           320
           ---
             0
```

```
(3.) 13)7.000000(.538461
        65·····
        --
         50
         39
         --
         110
         104
         ---
           60
           52
           --
            80
            78
            --
             20
             13
             --
             *7
```

```
(4.) 81)1.000000000(.012345679
        81·······
        --
        190
        162
        ---
         280
         243
         ---
          370
          324
          ---
           460
           405
           ---
            550
            486
            ---
             640
             567
             ---
              730
              729
              ---
               *1
```

In the *first*, a remainder 15, expressed in tens, hundreds, thousands, &c. is divided by 64, and the quotient .234375, therefore, expresses decimally the value of the fraction $\frac{15}{64}$; in the *second* example, it is necessary to annex ciphers before the division begins, and consequently the result .0368 represents the fraction $\frac{23}{625}$.

In both these examples, the operation terminates; but it will oftener happen, in the progress of the division, that the same remainder again emerges, after which the figures in the quotient must evidently maintain a perpetual recurrence. Thus, in the *third* example, the remainders of the division by 13 are successively 5, 11, 6, 8, 2, and again 7; from which point the series will recommence. The fraction $\frac{7}{13}$ is therefore, when expanded into decimals .538461, 538461, 538461, &c. continued in perpetual circulation.—In the *fourth* example, the remainders are 19, 28, 37, 46, 55, 64, 73, and then 1 as at first; here consequently a recurrence takes place, and the value of the fraction $\frac{1}{81}$ is expressed in these circulating decimals .012345679,012345679,012345679, &c. It deserves notice, that the two digits of those last remainders always make up 10.

In every case, the number of different remainders, and consequently the variety of changes, must obviously be fewer than the units contained in the divisor. The last example is extremely remarkable, since it brings out all the digits in their natural succession, except 8. The reason of such a curious result is, that $\frac{1}{81}$, being the square of

$\frac{1}{9}$, which, expanded into decimals, is .11111111, &c. must be determined by the multiplication here performed. The value of the fraction $\frac{1}{81}$, is, therefore, expressed by the successive figures 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, &c. continued for ever, through all the descending places

```
.1111111111, &c.
.1111111111, &c.
----------------
.0111111111
   111111111
    11111111
     1111111
      111111
       11111
        1111
         111
          11
           1
----------------
.01234567901 &c.
```

of the *Denary Scale;* but, at the junction of every *ten*, the order of the digits is partially disturbed; for the 11 behind the 10 changes it into 11, and this changes the 9 into 10, which again, by its influence, converts the 8 into 9.

The enlarged Multiplication Table, it is evident, might be applied with advantage to expedite the more complex operations in Division. Not to dwell on this subject, it will perhaps be sufficient to select an example of the contracted process employed when decimals are concerned.

Let it be required to find the tangent of 32°, which is performed by dividing the sine by the cosine of that arc.

The first member of the quotient here is 62, which, multiplied into 84, 80 and 48, give 5208, 4960 and 298, receding two places at each step. The remainder is 41295, which gives the second member 48 of the quotient. The rest of the operation proceeds in the same manner, excluding all the inferior terms.

```
.8480481).5299193(.6248694
          5208
            4960
              298
            -----
            41295
            4032
              384
                2
            -----
              589
              580
                6
            -----
                3
                3
```

As another instance of the management of complex quantities, suppose it were sought to divide the sexagesimal fraction 29°43′44″12‴ by 33°4′35″21‴.

33°4′35″21‴)29°43′44″12‴(53°55′39″30″

```
29  9
     3 32
       30 55
          19
-----------
   30 40 58
   30 15
      3 40
         32
-----------
      21 46
      21 27
          3
-----------
         16
         16
-----------
```

The operation will proceed thus: The first quotient 53°, being multiplied into the successive terms of the divisor, give 29°9′, 3′32″, 30″56‴ and 19‴; which, collected together, and subtracted from the dividend, leave 30′40″58‴, to be again divided. In this way, the descending process is renewed till the residue becomes extinguished, all the products below thirds being excluded.—This example exhibits the division of the difference of the sines of 42° and 100°, or of the cosines of the double arcs 48° and 80°, by twice the sine of 16°, and the quotient is consequently the sine of 64° expressed sexagesimally.

---

I have thus endeavoured to explain, with ample detail, the principles and various practice of Addition, Subtraction, Multiplication, and Division, which comprehend all the operations that strictly belong to Arithmetic. But the Extraction of Roots, though founded on a more abstruse analysis, yet comes under its range. The method of finding the Square Root has been already investigated, in treating of Palpable Arithmetic. But it may be instructive to resume the subject, and to extend the research to the Extraction of the Cube Root.

The evolution of roots can only approach by successive steps to the final result, descending from the highest to the lowest point of the given scale, by the series of additions required to complete the number sought. To discover the principle which should direct this operation, it is necessary to examine what must take place in the process of Compound Multiplication. The square of a number composed of two parts, will obviously consist of their four binary products; that is, it will include the square of each of those parts, together with their double product, as reciprocally multiplier and multiplicand. Taking away, therefore, the square of one of those parts, suppose the greater, there must remain the square of the other, joined to their double product; or, what is the same thing, this residue will be the product of the smaller, into a number formed by annexing it to double the greater. Consequently, to discover the secondary or additive portion of the root, we have only, after the square of the principal part has been separated, to divide what is left, by twice its root, annexing always the quotient to this divisor, in closing the process of division. The same operation, descending successively to lower terms, must be repeated, till the number proposed for extraction be entirely exhausted. It is only requisite to observe the rank and number of the figures which the root should contain. But, for this purpose, since every compound number will evidently by squaring have its places of figures doubled, we need only distinguish each pair in the number whose root is sought.

An example will best explain the whole procedure. Suppose it were required to find the square root of

107584. Beginning at the right hand, and marking every second figure, it is divided into three periods; which shows that the root must consist of hundreds, tens, and units. To the first period ten, the nearest square is 9, whose root 3 must occupy the place of hundreds. Subtracting and taking down the next period, the residue 175 comes to be decomposed; doubling, therefore, the root, we have 6 in the place of hundreds, to which the quotient 2, as denoting *twenty*, is annexed, and the product 124 set down for subtraction. The remainder, with the last period, making 5184, is finally to be analysed. Twice the root already found, amounting to 640, with the quotient 8 itself, forms the new divisor, and the product extinguishes what was left of the proposed number. The root is therefore 300, with the successive additions of 20 and of 8.

$$
\begin{array}{rl}
 & \dot{1}0\dot{7}5\dot{8}4(328 \\
 & 9 \\
\hline
62 & 175 \\
2 & 124 \\
\hline
648 & 5184 \\
8 & 5184 \\
\hline
 & 0
\end{array}
$$

It will sometimes be convenient, in performing this operation, to employ deficient figures, especially as they will rectify the oversight, in case too large a quotient may have been assumed.

$$
\begin{array}{rl}
 & \dot{1}1\dot{\bar{2}}\bar{4}\dot{\bar{2}}4(33\bar{2} \\
 & 9 \\
\hline
63 & 2\bar{2}\bar{4} \\
3 & 189 \\
\hline
66\bar{2} & \bar{1}\bar{3}\bar{2}4 \\
\bar{2} & \bar{1}\bar{3}\bar{2}4 \\
\hline
\end{array}
$$

We subjoin an example of the extraction of the same root on the *Duodenary Scale*. This practice will often be found useful in mensuration.

$$
\begin{array}{rl}
 & \dot{5}2\dot{3}1\dot{4}(234 \\
 & 4 \\
\hline
43 & 123 \\
3 & 109 \\
\hline
464 & 1614 \\
4 & 1614 \\
\hline
 & 0
\end{array}
$$

But to elucidate this subject fully, we shall likewise exhibit the same extraction carried through the inferior scales.

```
          BINARY.
  11010010001000000(10100100
  1
1001)1010
   1)1001
  1010001)1010001
        1)1010001
               000000

          TERNARY.
  1211012012 1(1100
  1
 21)21
  1)21
 22001)101201
     1) 22001
 220021)220021
      1)220021

          QUATERNARY.
  122101000(11020
  1
 21)22
  1)21
 2202)11010
    2)11010
          00

          QUINARY.
  11420314(2303
  4
 43)242
  3)23
 10103)30314
     3)30314
```

The same mode of proceeding will obviously extend to the descending terms of any scale, a pair of zeros being annexed to the remainder after each successive division. As an example, we shall select the calculation of the greater segment of a line, divided by extreme and mean ratio, which is found by subtracting *one-half* from the square root of *five-fourths*, the whole line being unit. The radical is thus determined on three different scales.

```
     QUATERNARY.
  1.10(1.0132032
  1
201)1000
  1) 201
2023)13300
   3)12201
20322)103300
    2)101310
2033003)13300000
      3)12231021
20330122)100231300
       2)101320310

     DENARY.
  1.25(1.118034
  1
 21)25
  1)21
 221)400
   1)221
 2228)17900
    8)17824
 223603)760000
      3)670809
 2236064)8919100
       4)8944256

     DUODENARY.
  1.30(1.14007
  1
 21)30
  1)21
 224)000
   4)894
 2280)22800
    0)20621
 22910)219000
     0)206491
 229207)1362000
      7)1374621
```

It hence appears, that the greater segment of a line, divided by extreme and mean ratio, is expressed in duodecimals by .74007, or extremely nearly by .75; and, therefore, that it consists of 89 parts, of which the whole contains 144. The very same result is obtained from the recurring series, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, &c. which continually approximates to the required value.

The mode which the Arabians used for extracting the square root very similar to their process of division, has been adopted by the Hindus. The example annexed will show the form of operation. Vertical lines being drawn, and the numbers distinguished into periods of two figures, the nearest root of 10 is 3, which is placed both below and above, and its square 9 subtracted; the 3 is now doubled, and 6 being written in the next column, is contained twice in 17, or the remainder with the first figure of the next period; the 2 is therefore set down both above and below, and being multiplied into 6 gives 12, which is subtracted from 17, leaving 5; the square of 2 or 4 is now subtracted from 55, and 518, the remainder with the succeeding figure, is divided by 64, or the double of 32, giving 8 for the quotient; then 8 times 64 is 512, which leaves 6 on being subtracted from 518; and 64 is exhausted, by taking from it the square of 8.

| | 3 | | 2 | | 8 |
|---|---|---|---|---|---|
| 1 | 0 | 7 | 5 | 8 | 4 |
| | 9 | | | | |
| | 1 | 7 | | | |
| | 1 | 2 | | | |
| | | 5 | 5 | | |
| | | | 4 | | |
| | | 5 | 1 | 8 | |
| | | 5 | 1 | 2 | |
| | | | | 6 | 4 |
| | | | | 6 | 4 |
| | | | | 4 | 8 |
| | | 6 | 6 | | |
| | 3 | | 2 | | |

Similar to this method, but less regular and systematic, was the process by which the earlier mathematicians of

Europe performed the extracting of the square root. I have annexed here an example of the operation, taken from the tract already mentioned of Regiomontanus. The question is, to find the square root of the number 5261216896. Now, the nearest square to 52 is 49, leaving 3 to be set above the 2, while 7, the root, is placed in the vertical line; then double of 7 or 14, being set under the 36, is contained twice, and 2 is accordingly placed under the 7; but twice 1 is 2, which, taken from 3, leaves 1; and twice 4 is 8, which, taken from 6 or 16, leaves 8, and extinguishes the 1 before it; and twice 2 itself is 4, which, taken from 1 or 11, leaves 7, and converts the preceding 8 into 7. In this way the process advances, till the figures become successively effaced. The root 72534 is placed both at the right hand side, and immediately below the work.

```
       123       |
      1465       |
    1757174      | 7
   38796595      | 2
  5261216896     | 5
     14406       | 3
      430        | 4
      145        |
       14        |
        1        |
  ---------------
     72534
```

The enlarged Multiplication Table will greatly facilitate the extraction of roots. As an example, let it be required to find the square root of 389061567504. This large number being distinguished into periods of four figures, the nearest square to the first is 3844, whose root is 62, of which the double 124 is contained 37 times in the remainder 466156, with the next period omitting the two last digits. In the same manner, is the process carried through the next step.

```
             .   .   .
          389061567504(623748
          3844
          ====
  12437 ) 466156
     37 ) 37
           888
            1369
            ====
1247448 ) 59877504
     48 ) 48
           1152
             3552
              12304
           ========
```

As other examples, I shall select the decimal extraction of the square root of 10, and of the fraction three-fourths.

$$\begin{array}{rl}
 & 1000(3.1622777 \\
 & 961 \\
6262 & 390000 \\
62 & 3844 \\
 & 3844 \\
632427 & 17560000 \\
27 & 1701 \\
 & 648 \\
 & 729 \\
63245477 & 4844710000 \\
77 & 4851 \\
 & 1848 \\
 & 4235 \\
 & 5929
\end{array}
\qquad
\begin{array}{rl}
 & .7500(.8660254038 \\
 & 7396 \\
17260 & 1040000 \\
60 & 102 \\
 & 15600 \\
1732025 & 44000000 \\
25 & 425 \\
 & 800625 \\
173205040 & 6993750000 \\
40 & 680 \\
 & 128201600 \\
17320508038 & 655484000000 \\
38 & 646 \\
 & 1216 \\
 & 19304 \\
 & 1444
\end{array}$$

A final example may be drawn from sexagesimals. Thus, suppose the square root were sought of $13024' \; 55'' \; 20''' \; 26^{iv} \; 10^{v} \; 37^{vi}$, which is the product of the radius into $217° \; 4' \; 55'' \; 20''' \; 26^{iv} \; 10^{v} \; 37^{vi}$, or the sum of 120°, or the diameter, and of the chord of 108°, or that of the tripled side of the inscribed decagon. The operation is performed as here shown, the nearest square to 13024′ being 12996′, of

$$\begin{array}{rllllll}
 & 13024' & 55'' & 20''' & 26^{iv} & 10^{v} & 37^{vi}(114°\,7'\,36''\,25''' \\
 & 12996 \\
228°\;7' & 28 & 55 & 20 \\
7' & 26 & 36 & 49 \\
228°\;14'\;36'' & 2 & 18 & 31 & 26 & 10 \\
36'' & 2 & 16 & 56 & 45 & 36 \\
228°\;15'\;12''\;25''' & 1 & 34 & 40 & 34 & 37 & 00 \\
25''' & 1 & 35 & 6 & 28 & 10 & 25
\end{array}$$

which the root is 114°, leaving 28 with the succeeding parts for the next decomposition. The result is 114° 7′ 36″ 25‴, which must therefore express sexagesimally the chord of the mean arc, or of 114°.

The cube, or the third power of a number composed of two parts, being the result of its triplicate multiplication, will evidently consist of eight members, which are the ternary products of its two components. These elementary products must hence include the cube of each part, and the three separate results of the square of each multiplied every way into the other, whether before the square or after it, or interposed between its sides. Consequently, the cube of a compound number is made up of the cubes of its two parts, together with thrice the product of the square of the first into the second, and thrice the product of the square of the second into the first. *To decompose a cube, therefore, after having subtracted the cube of the greater portion of its root, the remainder is to be divided by thrice the square of this portion for a quotient which approximates to the second portion of the root: The operation, however, is not completed till the divisor be enlarged by the addition of thrice the product of the first portion into the second, and by the square of the second. The complex divisor being now multiplied by the quotient and subtracted, may leave a remainder for a new process of decomposition.*— To pursue the operation, it is necessary previously to distinguish the number whose cube root is to be extracted into periods of three places of figures; and the ready application of the denary scale requires the annexed table of the cubes of the nine digits to be carefully committed to memory. The inspection of the figures shows that a cube, unlike a square number, terminates in all the digits indiscriminately. It may farther be remarked, that if a cube should end in 1, 4, 5, 6 or 9, its root will likewise end in the same figure; but if it terminate in any of the

| | |
|---|---|
| 1 | 1 |
| 2 | 8 |
| 3 | 27 |
| 4 | 64 |
| 5 | 125 |
| 6 | 216 |
| 7 | 343 |
| 8 | 512 |
| 9 | 729 |

remaining digits 2, 3, 7, or 8, the corresponding root will end in 8, 7, 3 or 2, that is, in the difference of each from 10. The final digit or cube, therefore, may always determine without ambiguity that of its root.

To exemplify the extraction of the cube root on the denary scale, I shall begin with a number consisting of only two periods. Here the nearest cube to 140, the first period, is 125, whose root is 5, the remainder with the next period is 15608, of which the approximate divisor is 75, the triple of the square of 5: Omitting the two last figures, it is contained *twice* in 156; then three times 5 or 15 multiplied by 2, and set one place lower, is 30, to which 4 or the square of 2 is annexed one place still lower. The descent of the figures is plain, since the 5 from its position has the value of 50, of which the triple square is 7500, and the triple product into 2 is 300.

```
75    | 140608(52
 30   | 125
   4  |  15608
7804  |  15608
```

I subjoin another example of greater extent. Of the first period 410, the nearest root is 7, triple the square of which is contained 4 times in 671, the remainder with the first figure of the next period; the additions to complete the quotient are three times 7, or 21 into 4, or 84 a place lower, and 16, the square of 4, another place below. The rest of the operation proceeds in the same way.

```
147      410172407(743
  84     343
    16 |  67172
15556  |  62224
16428    |  4948407
  666    |  4948407
     9   |
1649469
```

The large table of products will materially simplify and expedite the process of the extraction of the cube root. To adapt it the better for that purpose, I have likewise joined a table of the cubes of numbers as far as *one hundred.* A similar one of squares is likewise given, for the greater facility of consulting, though it was indeed contained already in the diagonal of the series of products. As an example, let it be required to extract decimally the cube root of *one-half.* Periods of six figures are now assumed; of the first, the nearest root is 79, and triple the square of this, or 18723, is found 37 times in 696100, the remainder with two zeros from the next period. The rest of the operation is easily understood.

```
                .500000(.7937
                 493039
18723           |6961
     8769       |666
         1369   | 2997
188108269       |    3034
                |      2553
                |  994047
```

The procedure which the Arabians and Persians followed in extracting the cube root, and by them communicated to the Hindus, resembles likewise their method of performing division. A short example may be judged sufficient for explaining this tedious plan of operation. Suppose it were required to find the cube root of 91125: Having drawn as many vertical lines as may be wanted, the several digits of the given number are inscribed between them, and dots set over the first, fourth, seventh, &c. reckoning from the right hand. Of the period 91, the nearest cube is obviously 64, which is set down and subtracted, leaving 27. To obtain the next figure, the triple of 16, the square of the root 4, is placed below, and being contained 5 times in 271, the divisor is now completed, by adding three times the product of 4 and 5, or 60, and then the square of 5 or 25, making in all 5425, each figure of which is multiplied by 5, and the products subtracted in succession.

| | 4 | | | 5 |
|---|---|---|---|---|
| 9 | 1 | 1 | 2 | 5 |
| 6 | 4 | | | |
| 2 | 7 | | | |
| 2 | 5 | | | |
| | 2 | | | |
| | 2 | 0 | | |
| | | 1 | | |
| | | 1 | 0 | |
| | | | 2 | |
| | | | 2 | 5 |
| | 4 | 8 | | |
| | | 6 | 0 | |
| | | | 2 | 5 |
| | 5 | 4 | 2 | 5 |

The ancient mode of extracting roots practised in

Europe was very similar to the process now explained, but not so regular and formal. The annexed example is copied from the *Ars Supputandi* of the famous Cuthbert Tonstall, Bishop of Durham; the earliest treatise of Arithmetic published in England, and a work indeed of no common merit. The number 25052358264 to be extracted is placed above two parallel lines, between which the root 6304 is inserted; the successive divisors and the corresponding remainders being written alternately below and above, and the figures erased as fast as the operation advances, the obliteration being here denoted by accents.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 4 | | 7 | 6 | | | | | | |
| | 3 | | 4 | 0 | | | | | | | |
| 2 | 5 | 0 | 5 | 2 | 3 | 5 | 8 | 2 | 4 | 6 | 4 |
| | | 6 | | | 3 | | | 0 | | | 4 |
| | 3 | 4 | 1 | 8 | 7 | 8 | 9 | 8 | 9 | 0 | 4 |
| | | 1 | 0 | 2 | 1 | 5 | 9 | | | | |
| | | | 1 | 2 | 2 | 5 | 1 | | | 0 | |
| | | | 4 | 9 | 6 | | 8 | 2 | 4 | 6 | |
| | | | | 7 | | | | | | | |

To leave nothing incomplete which relates to arithmetical operations, it will be proper likewise to investigate the nature and properties of *Vulgar Fractions*. It was observed in division, that, after the integral quotient has been found, the process may be still carried on through the descending bars of the numerical scale. This decomposition will either terminate after a few steps, or will repeat incessantly the same succession. But the subsidiary portion of the quotient might also be expressed as a *Fraction*, of which the *remainder* forms the *numerator*, and the *divisor* constitutes the *denominator*. In reference to quantity, therefore, the *numerator* shows how many units are to be collected, and the *denominator* indicates what share of this aggregate is to be assumed, for the value of the fraction.

If the numerator be increased, or the denominator diminished, the fraction must hence, in either case, be augmented; for the effect will evidently be the same, whether the quantity to be shared is enlarged, or the same quantity is to be divided into fewer portions. A fraction is consequently doubled by doubling the numerator, tripled by tripling it, and so forth; inversely again, the fraction is reduced to one-half by doubling the denominator, to one-third by tripling it, and so on continually. But, if both numerator and denominator be at the same time doubled, tripled, quadrupled, &c. the value of the fraction will not be altered, since any augmentation on the one hand is exactly counterbalanced by an equal and opposite diminution on the other.

On the principles now stated, most of the operations with fractions are grounded. A fraction, for instance, may be reduced to lower dimensions, if some number can be found that will divide both its numerator and denominator without a remainder. The number which is exactly contained in another is said to *measure* this; but if it measures more numbers than one it is called their *common measure.* To discover a common measure is, therefore, an important problem in the practice of fractions. The procedure is derived from the combination of two properties, which are almost evident from inspection: 1. *Any number that measures two others must likewise measure their sum and their difference*; and, 2. *A number that measures another, must measure also its multiple or its product by any integral number.* If two numbers be proposed, therefore, to find their common measure, it is only requisite to deduce from them other numbers successively smaller which shall contain likewise the same measure. Thus, suppose it were sought to reduce the fraction $\frac{77}{175}$ to lower terms. Whatever measures 77 must measure its double 154, or the nearest multiple to 175. Consequently it must also measure the difference of these, or 21. But it must measure 63, or the nearest multiple of 21 to 77. This common measure is hence contained in the difference of 63 and 77 or 14; but being contained likewise in 21, it must measure their difference or 7. Wherefore the common measure desired divides 7 without a remainder; but it divides also 14, which contains this 7, and is consequently 7 itself. Now 77 and 175, being divided by 7, give 11 and 25; and the fraction $\frac{77}{175}$ is thus reduced to $\frac{11}{25}$, which exhibits the same value in the lowest terms, for no number greater than 7 can fulfil the requisite condition of dividing 7.

Hence the process of finding the greatest common measure consists in the successive decomposition of the numbers proposed. Here 175 is divided by 77, and 77 by the remainder 21; next 21 is divided by the excess 14, and 14 itself finally by 7.

```
77)175(2
   154
    21)77(3
       63
       14)21(1
          14
           7)14(2
             14
```

In like manner, it is found by repeated division, that the greatest common measure of 748 and 2761 is 11. For this measure must be contained in three times 748 or 2144, and consequently in the difference of 2144 and 2761, or 517; and again in the difference of 507 and 748, or 231. Wherefore the number sought

```
748)2761(3
    2144
     517)748(1
         517
         231)517(2
             462
              55)231(4
                 220
                  11)55(5
                     55
```

must divide the double of 231 or 462, and the difference between this and 517, or 55. But measuring 55, it likewise measures 220, or its quadruple, and the difference of this from 231, or 11. It finally, therefore, divides both 11 and 55, and is hence 11 itself. The fraction $\frac{748}{2761}$, exhibited in its lowest terms, is thus $\frac{68}{251}$.

If a fraction be not capable of any reduction, this process of repeated analysis will always terminate in unit, as the only final divisor. But from the series of quotients may be derived another set of lower fractions, which will gradually approximate to the true value. To discover these subordinate fractions is a subject of curious and important investigation. For the sake, however, of distinctness and precision, it is here expedient to adopt a few of the simpler characters employed by Algebraists. The quotient of one number by another is readily expressed, by giving them a fractional form, the dividend and divisor constituting respectively the numerator and denominator. A full point prefixed to a number may indicate its multiplying that which precedes it; and the signs + and — will intimate the addition and subtraction of the numbers before which they are placed.

Let it be required now to decompound the fraction $\frac{108}{149}$. The value will not be altered, by dividing both numerator and denominator by the same number. Assume 27, which is contained in the numerator 108, and it will give the compound quotient $5\frac{14}{27}$, for the division of the denominator 149. The fraction may consequently be written thus, $\frac{4}{5\frac{14}{27}}$, or $\frac{4}{5+\frac{14}{27}}$. Again, by dividing the numerator and denominator of the appended fraction $\frac{14}{27}$ by 7, it will be changed into $\dfrac{2}{3\frac{6}{7}}$, or $\dfrac{2}{3+\frac{6}{7}}$. Wherefore, collecting these fragments into which the original fraction was broken, it will assume this compound form,

$$\cfrac{4}{5+\cfrac{2}{3+\cfrac{6}{7}}};$$

meaning that $\frac{6}{7}$ joined to 3 is to divide 2, and the quotient then added to 5 for a divisor of 4. If the extension of the lines which separate the numerators from their complex denominators be omitted, the fraction will merely run in a diagonal direction; thus,

$$\frac{4}{5+}\frac{2}{3+}\frac{6}{7}.$$

Fractions of this kind have, from the nature of their composition, been called *Continued Fractions.* They were first proposed about the year 1670 by Lord Brounker, President of the Royal Society, and improved by Dr Wallis, but afterwards cultivated chiefly on the Continent. Few speculations have proved finer in theory, or led to a more useful or prolific disclosure of the relations and properties of numbers.

The different parts of a continued fraction may, by an inverted process, be recombined and consolidated. Thus, the fraction

$$\frac{4}{5+}\frac{2}{3+}\frac{6}{7}$$

is again restored, by beginning at the

extreme portion $\frac{2}{3\frac{6}{7}}$, which being multiplied by 7, gives $\frac{14}{27}$, and then the complex fraction $\frac{4}{5\frac{14}{27}}$ is finally converted by the multiplication of its numerator and denominator by 27 into $\frac{108}{149}$. But if the terms of a continued fraction be taken in succession, other subordinate fractions will arise, which approach at each step to the true value. Thus, assuming only the two first terms $\frac{4}{5}+\frac{2}{3}$, it forms the fraction $\frac{12}{17}$; now the consecutive fractions $\frac{4}{5}$, $\frac{12}{17}$, and $\frac{108}{149}$, compose evidently an approximating series.

It is more convenient, however, to derive these secondary fractions by a direct succession, than to combine them by help of a retrograde procedure. Nothing in fact is wanted, but to reconsider the several multiplications that took place. The fractions as they are formed therefore proceed thus; $\frac{4}{5}$, $\frac{4.3}{5.3+2}$, and $\frac{4.3.7+4.6}{5.3.7+2.7+5.6}$. The first fraction $\frac{4}{5}$ has therefore both its numerator and denominator multiplied by 3, the denominator of the next fraction, and 2, the denominator of this fraction, added to the product of the denominator. Both numerator and denominator of this new fraction are next multiplied by 7, the denominator of the extreme fraction; and the products of the numerator and denominator of the preceding fraction multiplied by the numerator 6 are added. The symmetry of the process will appear more distinctly, if the fictitious fraction $\frac{0}{1}$ be placed at the beginning of the series:

Thus, $\frac{0}{1}$, $\frac{4}{5}$, $\frac{4.3+0.2}{5.3+1.2}$ or $\frac{12}{17}$, and $\frac{12.7+4.6}{17.7+5.6}$ or $\frac{108}{149}$. Hence, if the numerators of a continued fraction be all of them units, the recomposition is effected by multiplying with the successive denominators, and merely adding the numerator and denominator of the preceding fraction. This simpler kind of continued fractions, being the most common and convenient, the mode of transforming them deserves a separate investigation.

Suppose it were sought to decompose the fraction $\frac{287}{992}$. Divide both numerator and denominator by 287, and the complex fraction $\frac{1}{3\frac{131}{287}}$ arises, In like manner, divide $\frac{131}{287}$ by its numerator 131, and it is changed into the fraction $\frac{1}{2\frac{25}{131}}$. Again, reduce the fraction $\frac{25}{131}$, by dividing its terms by 25, and it assumes the complex form $\frac{1}{5\frac{6}{25}}$. Lastly, the fraction $\frac{6}{25}$ is converted into $\frac{1}{4\frac{1}{6}}$. Wherefore, introducing those substitutions, the original fraction will exhibit these successive phases:

I. $\frac{287}{992}$; II. $\frac{1}{3+\frac{131}{287}}$; III. $\cfrac{1}{3+\cfrac{1}{2+\frac{25}{131}}}$; IV. $\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{5+\frac{6}{25}}}}$;

V. $\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{6+\cfrac{1}{4}}}}}$; Or more simply thus, without the prolonged lines, $\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{5+}\,\frac{1}{6+}\,\frac{1}{4}$.

It is obvious, that this process of decomposition is conducted precisely in the same way as the operation for finding a common measure. The successive quotients, 3, 2, 5, 4, and 6, here become the denominators of the continued fraction.

```
287)992(3
    861
    131)287(2
        262
         25)131(5
            125
              6)25(4
                24
                 1)6(6
```

To restore a continued fraction again, the most obvious mode is, beginning at its extreme term, to re-ascend by successive combinations. Thus, resuming the same example, $\frac{1}{6\frac{1}{4}}$ is first changed into $\frac{6}{25}$, then $\frac{1}{5\frac{6}{25}}$ passes into $\frac{25}{131}$; next $\frac{1}{2\frac{25}{131}}$ is condensed into $\frac{131}{287}$; and lastly, $\frac{1}{3\frac{131}{287}}$ is transformed into $\frac{287}{992}$.

By commencing this process of consolidation from any intermediate step, a series of subordinate and approximative fractions will be likewise obtained. Thus, $\frac{1}{3\frac{1}{2}}$ gives $\frac{2}{7}$, and of the three terms,

$$\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{5}}}$$

first $\frac{1}{2\frac{1}{5}}$ is equal to $\frac{5}{11}$, and $\frac{1}{3\frac{5}{11}}$ equal to $\frac{11}{38}$, and then $\frac{1}{3\frac{5}{11}}$ is condensed into $\frac{11}{38}$; again, the four terms,

$$\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{4}}}}$$

give first $\frac{1}{5\frac{1}{4}}$ or $\frac{4}{21}$, then $\frac{1}{2\frac{4}{21}}$ make $\frac{21}{46}$; and lastly, $\frac{1}{3\frac{21}{46}}$ is equal $\frac{46}{159}$. Thus, the following series of approximative fractions has been formed, $\frac{1}{3}$, $\frac{2}{7}$, $\frac{11}{38}$, $\frac{46}{159}$, $\frac{287}{992}$, commencing with $\frac{1}{3}$, and terminating in $\frac{287}{992}$.

If the first of these fractions have its numerator and denominator multiplied by 7, and the second, in like manner, by 3, they will be changed into the equivalent fractions $\frac{7}{21}$ and $\frac{6}{21}$; whence the value decreases in the second by $\frac{1}{27}$. But let the next adjacent pair of fractions, $\frac{2}{7}$ and $\frac{11}{38}$, have their numerators and denominators multiplied by 38 and 7, and they will become $\frac{76}{266}$ and $\frac{77}{266}$; so that the value increases at the third fraction by the difference $\frac{1}{266}$. In the same way, it is found that the fraction $\frac{46}{159}$ suffers a diminution of $\frac{1}{6042}$, and the original fraction $\frac{287}{992}$ receives an augmentation of $\frac{1}{157728}$. The series of subordinate fractions thus alternately oscillate above and below the true value, to which, however, they rapidly approach. Hence the original fraction might likewise be expressed, by combining those alternating differences: $\frac{1}{3}-\frac{1}{21}+\frac{1}{266}-\frac{1}{6042}+\frac{1}{157728}$. The gradual approximation is, therefore, clearly marked, since these successive fractions have only *one* for their numerator.

It would be more commodious to discover those subordinate fractions by a direct procedure. Let the continued fraction $\frac{1}{3+\frac{1}{2+\frac{1}{5+\frac{1}{4+\frac{1}{6}}}}}$ be resumed. The progressive steps are thus exhibited: I. $\frac{1}{3}$. II. $\frac{1}{3\frac{1}{2}}$, or $\frac{1.2}{3.2+1}=\frac{2}{7}$.

III. $\frac{1}{3+\frac{1}{2\frac{1}{5}}}$; instead of 2 in the preceding expression, substitute $2\frac{1}{5}$, and the result is $\frac{1.2\frac{1}{5}}{3.2\frac{1}{5}+1}=\frac{1.2.5+1}{3.2.5+5.1+3}=$ $\frac{2.5+1}{7.5+3}=\frac{11}{38}$. IV. $\frac{1}{3+\frac{1}{2+\frac{1}{5\frac{1}{4}}}}$; instead of 5 in the last expression substitute $5\frac{1}{4}$, and there comes out $\frac{2.5\frac{1}{4}+1}{7.5\frac{1}{4}+3}=$ $\frac{2.5.4+4+2}{7.5.4+3.4+7}=\frac{11.4+2}{38.4+7}=\frac{46}{159}$; and finally,

V. $\frac{1}{3+\frac{1}{2+\frac{1}{5+\frac{1}{4\frac{1}{6}}}}}$; substitute $4\frac{1}{6}$ for 4 in the preceding, and it becomes changed into $\frac{11.4\frac{1}{6}+2}{38.4\frac{1}{6}+7}=\frac{11.4.6+2.6+11}{38.4.6+7.6+38}=$ $\frac{46.6+11}{159.6+38}=\frac{287}{992}$.

The order of succession is hence easily perceived. If the simulated fraction $\frac{0}{1}$ be placed first, the approximative fractions will be formed, by constantly multiplying the series of denominators, and adding the preceding terms: Thus, $\frac{0}{1}, \frac{1}{3}, \frac{2}{7}, \frac{11}{38}, \frac{46}{159}, \frac{287}{992}$.

It may likewise be shown, that the alternate products of the numerators and denominators of these fractions differ incessantly in excess and defect by unit: Thus, $3.2+1=7.1$, $7.11-1=38.2$, $38.46+1=159.11$, and $159.287-1=992.46$. For resuming the former analysis, $7=3.2+1$; $7.11=7.2.5+7$ and $38.2=7.5.2+3.2$; $38.46=38.11.4+38.2$, and $159.11=38.4.11+7.11$; and finally, $159.287=159.46.6+159.11$, and $992.46=159.6.46+38.46$. This mode of decomposition, though employed here in a particular example, is evidently quite general.

These approximative fractions hence will terminate always in the true value. Any fraction may therefore be reduced to its lowest term directly from the series of quotients evolved in finding the common measure of its numerator and denominator. Thus, the fraction $\frac{77}{175}$ gave the quotients 2, 3, 1, 2, and consequently is converted into the continued fraction

$$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2}}}};$$

of which the approximating values are $\frac{0}{1}, \frac{1}{2}, \frac{3}{7}, \frac{4}{9}$, and $\frac{11}{25}$, the last being the reduced fraction. Again, the fraction $\frac{748}{2761}$ furnished these quotients, 3, 1, 2, 4, and 5. It may therefore be changed into the continued fraction $\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{5.}}}}}$

But this again will produce the approximative fractions, $\frac{0}{1}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{3}{11}$, $\frac{13}{48}$, and $\frac{68}{257}$, the last being only the original fraction reduced to its lowest terms.

The secondary fractions derived from the successive composition of the members of a continued fraction, it has been shown, differ from each other by alternating variations, which have always unit as their numerator. Such derivative fractions will, therefore, not only advance rapidly to the true expression, but must constantly approach the nearest possible, or exhibit the approximate value in the lowest terms. This property is of essential consequence in all operations with numbers, and furnishes many useful practical results. A few examples will justify the remark.

Let it be required to express approximately the fractional portion of 24 hours, by which the solar year exceeds 365 days. This excess, or 5 hours 48 minutes and 50 seconds, being reduced to seconds, makes 20930″, while 24 hours give 86400″. Wherefore the fraction $\frac{20930}{86400}$, or $\frac{2093}{8640}$, is to be decomposed.

```
2093)8640(4
     8372
      268)2093(7
          1876
           217)268(1
               217
                51)217(4
                   204
                    13)51(4
                       52
```

The successive quotients are 4, 7, 1, 4, and 4, without pushing the last division with rigour. There results, consequently, the continued fraction

$$\cfrac{1}{4+\cfrac{1}{7+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{4}}}}};$$

from which are derived the approximative fractions $\frac{1}{4}$, $\frac{7}{29}$, $\frac{8}{33}$, $\frac{39}{161}$, and $\frac{164}{677}$.

Some of these fractions are remarkable. The first fraction $\frac{1}{4}$ indicates the insertion of one day every four years, being the correction of the Kalendar by the *Bissextile* or *Leap Year* introduced by Julius Cæsar. The fraction $\frac{8}{33}$ indicates a more correct intercalation of 8 days in 33 years; a method proposed about six centuries ago by the Persian Astronomers, who, after the lapse of seven ordinary leap years, always deferred the eight return of the period one year longer than usual.—If this fraction $\frac{8}{33}$ had its numerator and denominator multiplied by 12, and those of $\frac{1}{4}$ added to the products, another fraction $\frac{97}{400}$ will be formed, of very nearly the same value. This last represents the intercalation established by Pope Gregory in 1582, and afterwards successively adopted by all the Christian powers except Russia, which affects to maintain the independence of the Greek Church. It implies the insertion of 97 days in the space of 400 years; which is performed by combining the Julian system with an omission of three intercalary days in four centuries; that is, the last year of each century, which falls to be a leap year, is not considered as such, unless the number of the century itself is divisible by four.—But the Persian mode of correcting the kalendar was evidently simpler and more elegant, since in the space of 33 years it restored the coincidence which we now require the course of 400 years to effect.

As another example, let it be required to express the English foot by the French *metre*, or unit of linear measures. The metre being 39.371 inches, gives, on a division by 12, the continued fraction.

$$\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{7.}}}}}}}}$$

Wherefore the approximating fractions are $\frac{1}{3}$, $\frac{3}{10}$, $\frac{4}{13}$, $\frac{7}{23}$, $\frac{25}{82}$, $\frac{32}{105}$, $\frac{89}{292}$, and $\frac{655}{2149}$. Hence the foot is to the metre nearly as 3 to 10, and still more accurately as 32 to 105.

As a farther illustration, it was found, in the extraction of roots, that the reciprocal of .7937 must express the side of the double cube, to discover which, without the powerful aid of the denary arithmetic, exercised the utmost ingenuity of the ancient Greeks. If that decimal in relation to unit were converted into a continued fraction, the quotients forming the series of denominators would be 1, 3, 1, 5, 1, 1, 4, 1, 1, 2, 1, 1 and 2. Wherefore the successive approximations are $\frac{0}{1}$, $\frac{1}{1}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{23}{29}$, $\frac{27}{34}$, $\frac{50}{63}$, $\frac{227}{286}$, $\frac{277}{349}$, $\frac{504}{635}$, &c. Of these fractions, one of the simplest and most remarkable is $\frac{50}{63}$; the cube of 63 being 250047, while that of 50 is 12500, very nearly the half. The expression $\frac{504}{635}$, however, approximates still nearer, the cube of 635 being 256047875, and that of 504 being 128024064.

Continued fractions will afford elegant approximations to the square root of any number. Thus, the square root of 10 is 3, together with the remainder 1, divided by 6 and this quotient itself. The fractional part, therefore, changes, by repeated substitution, into $\cfrac{1}{6+\cfrac{1}{6}}$, then into $\cfrac{1}{6+\cfrac{1}{6+\cfrac{1}{6}}}$, and thus passes into a continued fraction, which has likewise no termination. The square root of 10 is therefore $3+\cfrac{1}{6+\cfrac{1}{6+\cfrac{1}{6}}}$, &c. If the fractional part be successively combined, it will form this range of approximating fractions, of which the numerators and denominators evidently constitute a recurring series; $\frac{1}{6}$, $\frac{6}{37}$, $\frac{37}{228}$, $\frac{228}{1405}$, $\frac{1405}{8658}$, &c. The last of these converted into decimals gives 3.162277 for the square root of 10, being true to every place of figures. It is what the Arabians and Hindus sometimes most inaccurately assumed for the circumference of a circle whose diameter is 1, and which Joseph Scaliger afterwards pompously announced as a perfect quadrature.

The square root of 11 must be 3, with the remainder 2 divided by 6, and the quotient itself. Wherefore, by repeated substitutions, the root will be expanded into this continued fraction, $3+\cfrac{2}{6+\cfrac{2}{6+\cfrac{2}{6}}}$, &c., which shoots onwards without intermission. It may be reduced, however, to a simpler form, the expression for $\cfrac{2}{6+\cfrac{2}{6}}$ is evidently the same as $\cfrac{1}{3+\cfrac{1}{6}}$, the terms only being divided by 2. Whence the square root of 11 is $3+\cfrac{1}{3+\cfrac{1}{6+\cfrac{1}{3+\cfrac{1}{6}}}}$, &c., the circulation occurring at every second place.—In like manner, the square root of 12 is $3+\cfrac{3}{6+\cfrac{3}{6}}$, &c., which may be converted into $3+\cfrac{1}{2+\cfrac{1}{6}}$, &c., circulating also at every second place.

But the terms of a continued fraction, which expresses the square root of any number, are not always capable of being so easily reduced. Thus, the square root of 7 must be $2+\cfrac{3}{4+\cfrac{3}{4}}$, &c.; where the numerator, not being contained in the denominator of the repeating fraction, this will admit of no direct change. It may still be transformed, however, by a circuitous procedure. The fractional expression $\frac{3}{4+\frac{3}{4}}$, &c. gives, by reduction, this series of approximative fractions, $\frac{3}{4}$, $\frac{12}{19}$, $\frac{57}{88}$, $\frac{264}{409}$, $\frac{1227}{1900}$, &c. But the last of these, converted into a simple continued fraction, furnishes the denominators 1, 1, 1, 4, 1, 1, 4, &c., from which the circulation is easily inferred. Wherefore the square root of 7 is $2+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{4}}}}$, &c.; and consequently the fractional part is thus expressed by successive approximation $\frac{1}{1}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{9}{14}$, $\frac{11}{17}$, $\frac{20}{31}$, $\frac{31}{48}$, $\frac{144}{223}$, $\frac{175}{271}$, $\frac{319}{494}$, $\frac{494}{767}$, $\frac{2295}{3562}$, &c.

There is another method of decompounding fractions, which likewise deserves particular notice. An example will explain this procedure. Resuming the fraction $\frac{287}{992}$, it may be written $\frac{861}{3.992}$, or $\frac{992-131}{3.992}$, or more simply $\frac{1}{3}-\frac{131}{3.992}$: Again, the latter part $\frac{131}{992}$ of the compound fraction, is transformed into $\frac{917}{7.992}$, or $\frac{992-75}{7.992}$, or merely $\frac{1}{7}-\frac{75}{7.992}$: But the fraction $\frac{75}{992}$ is, in like manner, changed into $\frac{975}{13.992}$, or $\frac{992-17}{13.992}$, or $\frac{1}{13}-\frac{17}{13.992}$: Now, $\frac{17}{992}=\frac{986}{58.992}=\frac{992-6}{58.992}=\frac{1}{58}-\frac{6}{58.992}$: Again, $\frac{6}{992}=\frac{990}{165.992}=\frac{992-2}{165.992}=\frac{1}{165}-\frac{2}{165.992}$. And, lastly, $\frac{2}{992}=\frac{1}{496}$. Wherefore, collecting these elements and reascending, the original fraction $\frac{287}{992}$ is converted by substitution into the alternating series, $\frac{1}{3}-\frac{1}{3.7}+\frac{1}{3.7.13}-\frac{1}{3.7.13.58}+\frac{1}{3.7.13.58.165}-\frac{1}{3.7.13.58.165.496}$. This series, however, conveys rapidly, as will appear from the actual multiplication of the factors; thus, $\frac{1}{3}-\frac{1}{21}+\frac{1}{273}-\frac{1}{15834}+\frac{1}{2612610}-\frac{1}{120184060}$. It is obvious that the factors which compose the series representing the fraction $\frac{287}{992}$ are only the quotients of the continued division of the denominator 992, by the numerator 287, and the successive remainders. The operation is here given at length, the same dividend being constantly repeated.

```
287)992(3
    861
    131)992(7
        917
         75)992(13
            975
             17)992(58
                  &c. &c.
```

As another example, suppose it were required to convert the decimal expression 3.14159265 for the circumference of a circle whose diameter is 1, into a converging series: Dividing 1 by the part .14159265, and by the remainders which appear in succession, these quotients will result; 7,113,4739, and 47051, &c. Whence the circumference will be denoted by $3+\frac{1}{7}-\frac{1}{7.113}+\frac{1}{7.113.4739}-\frac{1}{7.113.4739.47051}+$ &c.

The first term of the series indicates the Archimedean proportion, and the conjunction of the next term gives that of Metius.—This ingenious mode of transforming fractions was the invention of the celebrated Lambert, a mathematician and philosopher distinguished by his strong and original powers of thinking, who left his native spot in the verge of the Swiss Cantons, and accepted the invitation of the great Frederic to settle at Berlin, where he died in 1777.

The other properties and combinations of fractions will be more easily discussed.

1. To convert fractions into others which are equivalent, but have a common denominator, it is only requisite to multiply the numerator and denominator of each fraction into the continued product of all the rest of the denominators. Thus, the fractions, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{5}$ and $\frac{4}{7}$ are changed into $\frac{1.3.5.7}{2.3.5.7}$, $\frac{2.2.5.7}{3.2.5.7}$, $\frac{3.2.3.7}{5.2.3.7}$ and $\frac{4.2.3.5}{7.2.3.5}$; that is, $\frac{105}{210}$, $\frac{140}{210}$, $\frac{126}{210}$, and $\frac{120}{210}$. The reason is obvious from the constitution of fractions.

2. To add or subtract fractions, nothing is wanted but to reduce them to the same denomination, and the addition or subtraction of their numerators will give as the result, new numerators to their common denominator. Thus, $\frac{3}{5}$, $\frac{2}{3}$, and $\frac{1}{7}$ are first reduced to $\frac{63}{105}$, $\frac{70}{105}$ and $\frac{15}{105}$, and

their sum is $\frac{148}{105}$ or $1\frac{43}{105}$. Again, $\frac{2}{5}$ subtracted from $\frac{3}{7}$ gives $\frac{15}{35}, -\frac{14}{35}$, or $\frac{1}{35}$.

3. To multiply fractions. It is evident that the fractional multiplier implies that the multiplicand is to be repeated as often as the former has units in its numeration, and then subdivided as often into as many parts as there are units in its denominator. Consequently, the numerator of the multiplicand should be multiplied by the numerator of the multiplier, and divided by its denominator; or, instead of this last operation, the result must be the same, if the denominator of the multiplicand be multiplied by the denominator of the multiplier. Thus, the fraction $\frac{2}{3}$ multiplied into $\frac{6}{7}$, signifies that $\frac{6}{7}$ is to be repeated twice, and the amount to be then divided into three equal portions; but $\frac{6}{7}$ being doubled, makes $\frac{12}{7}$, and this divided by 3 gives $\frac{4}{7}$. Instead, however, of dividing the 12 by 3, the effect is the same to triple the denominator 7, for $\frac{4}{7}$ and $\frac{12}{21}$ are evidently equivalent fractions. The division of the numerator being seldom practicable in whole numbers, the corresponding multiplication of the denominator is the general and preferable method. Hence, successive fractions are multiplied by multiplying all the numerators for a new numerator, and all the denominators for a new denominator.

Thus, the continued product of $\frac{3}{4}$, $\frac{5}{6}$, and $\frac{2}{7}$, is $\frac{30}{168}$, or $\frac{5}{28}$. Again, the product of the successive multiplication of $\frac{11}{2}$, $\frac{7}{3}$, and $\frac{8}{5}$, is $\frac{616}{30}$, or $20\frac{8}{15}$. This result might likewise in this case be found, though more laboriously, by the method of Cross Multiplication. The operation would proceed thus: $\frac{11}{2}$ and $\frac{7}{3}$ being the same as $5\frac{1}{2}$ and $2\frac{1}{3}$, the 2 is multiplied into 5, and next into $\frac{1}{2}$; then $\frac{1}{3}$ is multiplied into 5 and into $\frac{1}{2}$: The sum is $12\frac{5}{6}$, which being multiplied again by $\frac{8}{5}$, or by 1 and then by $\frac{3}{5}$, gives the total result as before.

$$\begin{array}{l}
5\ \frac{1}{2} \\
\underline{2\ \frac{1}{3}} \\
10\ 1 \\
\quad 1\frac{2}{3}\frac{1}{6} \\
\hline
12\frac{5}{6} \\
\underline{1\frac{3}{5}} \\
12\frac{5}{6} \\
7\frac{1}{5}\frac{1}{5}\frac{5}{6} \\
\hline
20\frac{8}{15}
\end{array}$$

4. The most obvious method of performing the division of fractions is to reduce them to a common denomination, and to state the quotient of their transformed numerators. Thus, to divide $\frac{3}{4}$ by $\frac{5}{7}$; they are first changed into the equivalent fractions $\frac{21}{28}$ and $\frac{20}{28}$, and now $\frac{20}{21}$ must express the quotient. The same result would be obtained if the divisor were converted into its reciprocal, and then multiplied; for the numerator and denominator being interchanged, must exactly reverse the nature of the operation. Thus, the reciprocal of $\frac{3}{4}$ or $\frac{4}{3}$ being multiplied into $\frac{5}{7}$ gives $\frac{20}{21}$, as derived before. The general rule, consequently, for the division of fractions, is to multiply crosswise the numerator of the divisor into the denominator of the dividend for a new denominator, and the denominator of the divisor into the numerator of the dividend for a new numerator.

Those fractions which are the reciprocals of integers, or have one for their numerator, exhibit, when transferred to the descending bars of any scale, either the simple repetition of unit, or a succession of digits expanding in uniform progression. It was already shown, that $\frac{1}{9}$, converted into a decimal expression, gives first 1, with a remainder of 1; and must therefore evolve in its subsequent expansion always the same train of units. If the fraction $\frac{1}{8}$ be represented on this scale, the first term will be 1, with 2 of a remainder; consequently, each succeeding term is constantly doubled, forming the series .1, .02, .004, .008, .00016, &c. Again, $\frac{1}{7}$ treated in this manner, gives first 1, with an excess of 3; and hence all the subsequent terms are successively tripled, composing the progression .1, .03, .009, .0027, .00081, &c. In like manner, since the quotient of 10 by 6 is 1, with the remainder 4, the fraction $\frac{1}{6}$, converted into decimals, will form the series .1, .04, .0016, .00064, 000256, &c. proceeding by constant quadruplication. It may, therefore, be generally inferred, that the reciprocal of any number is exhibited on a descending scale by a series of digits consisting of unit, followed by the excess of the index and its successive powers.

The descending progression, which represents the reciprocal of any number, though it never terminates, will sometimes converge to a definite result. Thus, the expansion of the fraction $\frac{1}{8}$, or the sum of the terms .1,2,4,8
16
32
64
128 &c.

being .12499968, receives always another 9 at each step of the progress, and consequently approximates to .125, the true decimal value. In other cases, the summation of the figures forms either a perpetual repetition or a periodic circulation. Thus, $\frac{1}{6}$ being expanded, gives .1,4
16
64
256
1024
4096,
which makes .1663936, but when pursued farther, approaches constantly nearer to the complete expression .166666, &c. Again, the fraction $\frac{1}{7}$ is expanded into the decimal terms .1,3,9
27
81
243
729, &c., which, being collected together, indicate the period .142857, 142857, 142857, &c. of incessant circulation.

If the number whose reciprocal is expressed on a descending scale be only less than some power of the index, the same progression of digits will emerge at corresponding intervals. Thus the decimal of $\frac{1}{98}$ is .01020408, the terms doubling at every second place. Hence $\frac{1}{7}$, which is the same as $\frac{14}{98}$, will be expressed by continually doubling 14, and descending two places. Accordingly, .14,28,56
112
224
448, &c. taken collectively, give .142857, in per-

petual circulation. It follows likewise, that $\frac{2}{7}$, being the double of this fraction, is expressed by the same series of figures, only commencing two places lower; thus, 285714, 285714, 285714, &c. In the same manner $\frac{4}{7}$ will yet be denoted by the same descending progress, beginning at the next binary period, or two digits still lower; thus, .571428, 571428, 571428, &c. But the fraction $\frac{3}{7}$ being equal to $\frac{42}{98}$, must, from the principle already stated, be composed by the summation of the decimal .42, redoubled at every second place, in this manner, .42,84

168
 336
 672, which makes up

.428571, 428571, 428571, &c. the same series again only repeating from the second term. Hence $\frac{6}{7}$, or the double of the last, will be represented by this identical series, only beginning two places lower, or at the fourth term; thus .857142, 857142, 857142, &c. Lastly, $\frac{5}{7}$ or $\frac{70}{98}$, being expressed by 70 and its successive reduplications, or merely by 7 redoubled at every second place, will be represented by 714285, 714285, 714285, &c. It thus appears, that the several multiples of 142857, as far as the product by six inclusive, exhibit always the same train of digits, and in the same order of succession. Seven times the same number gives 999999.

142857
285714
428571
571428
714285
857142

This property of the number 142857, which becomes always renascent in its multiples, may be deemed a singular arithmetical curiosity. It depends on the peculiar circulation of the decimal expression for the fraction $\frac{1}{7}$, the digits of which are not disturbed by the lower multiplications, because the denominator is less than 10.

When the reciprocal of any integer is expounded by a decimal circulation, the figures forming the period must evidently amount to less than the number itself, since the moment any remainder of the division is repeated, a cycle will commence. Any number will hence divide an integer, with a definite succession of ciphers annexed, or this quantity diminished by unit; and conversely, any periodic decimal may be converted into a vulgar fraction, of which the denominator consists of a certain repetition of *nines*. These observations are easily extended to other scales of Numeration. Enough, however, has perhaps been already stated in explication of the Theory of terminating, repeating, and circulating Decimals.

# NOTES

AND

# ILLUSTRATIONS.

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## Note I.—Pages 1—6.

PHILOSOPHERS, misled by the hasty and careless reports of travellers, have generally much underrated the attainments of savage tribes in the art of numeration. From the mere scantiness of the terms which a rude people employs to signify numbers, it would at least be rash to infer the narrow range of their application. The language even of the most polished nations, when traced to its radical form, is yet found to betray uncommon poverty in numerical expression. In the ancient Gothic, *Tachund*, or simply *Hund*, denoted only *Ten;* and the word *Hundred*, composed by annexing *Red* or *Ret*, the participle of the verb *Reitan*, *to reckon or place in rows*, intimated consequently that its *root was to be ten times told.* In the translation of the Gospels, which Bishop Ulphilas made for the use of the Goths in the fourth century, Fourcera, *One Hundred*, is expressed by *Tachund Tachund*, or the name for *Ten* merely redoubled. The word *Teon* had also become synonymous with *Hund*, and in the Anglo-Saxon version, compiled three centuries after that period, *One Hundred* is called *Hund Teontig*, or *ten ten times drawn.* The name for the next number on the Denary Scale, or *One Thousand*, is nothing but an abbreviation of *Diuis-Hund*, or *twice-ten*, meaning that *ten* was to be counted over in a double succession, making first *one hundred*, and then *one thousand.*

The ancient Scandinavians were fond of reckoning by *dozens*, and they sometimes combined to a certain extent the denary with the duodenary scale. The Chinese likewise employ both those scales in expressing their cycle of *sixty* years, which consists of *ten roots* and *twelve branches*, the year of the cycle being signified by the remainders in counting it by 10 and by 12. In like manner, ecclesiastical historians usually marked the dates of events by the numbers of the Solar and Lunar cycles, or reckoned by 28 and 19, which return again in the same order after a period of 532 years, called the Cycle of Indiction.

Bishop Tonstall, in his Arithmetic, aptly compares the extension given to numbers, by help of the denary system of classification, to the growth of reeds, which, though slender, are enabled to shoot to such a great height, from the joints interposed along their stalks. *Quemadmodum in arundinibus internodia paribus distincta spatiis, proceritatem ipsam producunt calamorum; sic in majoribus numeris alia aliis aggregata dena, velut numerorum internodia, crescentem magnitudinem connectunt.*

## Note II.—Pages 7—9.

The notation of numbers by combined strokes, which the Romans had received from their Grecian progenitors, was evidently founded in nature, and may be regarded as one of the earliest samples of a philosophical language. It is not surprising, therefore, that other nations, without supposing any communication, should have advanced by the same road.

That the Roman system of notation was originally formed by successively combining the simple strokes, derives strong confirmation from the analogous practice of other people. It appears, from obvious inspection, that the Egyptians and the Chinese must have followed nearly the same mode. The inscriptions on the ancient obelisks present a few numerals which are easily distinguished. Thus, the single stroke denotes *one*,

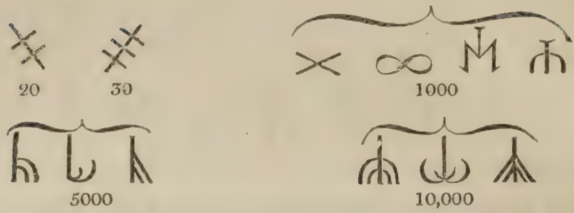

the St George's cross *ten*, and the half of it *five*, and the same cross doubled *an hundred*, a zero a *thousand*, and this zero combined with the marks for 5 and 10, signifies *five thousand* and *ten thousand*.

The substitution of capital letters for the combined strokes which they chanced most to resemble, gave uniformity indeed to the system of writing, but fatally prevented any farther improvements in numeral notation. The only simplification which the Romans appear to have introduced, was to avoid the profuse repetition of letters, by reckoning in some cases backwards. Thus, they denoted *Four* by IV, *Nine* by IX, *Forty* by XL, and *Ninety* by XC, &c.; which signified *five* and *ten*, *abating one from each*,—and *forty and an hundred*, *diminished each by ten.* The series of Roman numerals is thus exhibited:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| I. | II. | III. | IV. | V. | VI. | VII. | VIII. | IX. |
| 1. | 2. | 3. | 4. | 5. | 6. | 7. | 8. | 9. |
| X. | XX. | XXX. | XL. | L. | LX. | LXX. | LXXX. | XC. |
| 10. | 20. | 30. | 40. | 50. | 60. | 70. | 80. | 90. |
| C. | CC. | CCC. | CCCC. | D. | DC. | DCC. | DCCC. | CM. |
| 100. | 200. | 300. | 400. | 500. | 600. | 700. | 800. | 900. |

| | | | | | |
|---|---|---|---|---|---|
| D or IↃ· | M or CIↃ· | IↃↃ· | CCIↃↃ· | IↃↃↃ· | CCCIↃↃↃ· |
| 500. | 1000. | 5000. | 10,000. | 50,000. | 100,000. |

In illustration, it may be observed, that Cicero has this expression in his fifth oration against Verres: CIↃ CIↃ CIↃ IↃC. meaning 3600. The Romans, however, sometimes contracted or modified the forms of their numerals. This was done for the sake of expedition, chiefly in the carving of inscriptions on stones; and the abbreviated letters then used were called *lapidary characters*. The annexed specimen shows the principal varieties:




The marks for any number could also be augmented in power *one thousand times*, either by inclosing them with two hooks or C's, or by drawing a line over them. Thus, CXↃ, or $\overline{\text{X}}$, denoted 10,000; and $\overline{\text{CLVIM}}$ in Pliny means 156,000,000. Sometimes a smaller letter was placed above another to signify their product; thus $\overset{\text{D}}{\text{M}}$ would express 50,000. Or the multiplier was written like an exponent at the upper corner; thus $\text{III}^{\text{c}}$ was only another mode of signifying *three hundred.* In expressing very large numbers, points were sometimes interposed—a practice which, had it become more general, would have effected a material improvement. Thus, Pliny denotes 1,620,829 by these divided characters, XVI. XX. DCCC XXIX.

But the Romans appear, in the latter ages of their Empire, to have likewise employed the small letters of the alphabet, in imitation of the numeral system of the Greeks. The letters a, b, c, d, e, f, g, h, and i, represented the nine digits, 1, 2, 3, 4, 5, 6, 7, 8, and 9; the next series, k, l, m, n, o, p, q, r, and s, expressed 10, 20, 30, 40, 50, 60, 70, 80, and 90; and the remaining letters t, u, x, y, and z, denoted 100, 200, 300, 400, and 500. To exhibit the rest of the centuries, it was requisite to borrow capitals or other characters, and 600, 700, 800, and 900, were accordingly represented by I, V, hi and hu. But this mode of notation never obtained any degree of currency, being mostly confined to those foreign adventurers from Greece, Egypt or Chaldæa, who, by pretending to skill in judicial astrology, were enabled to prey on the credulity of the wealthy Romans.

In modern Europe the Roman numerals were supplied by Saxon characters. Thus, in the accounts of the Scottish

Exchequer for the year 1331, the sum of L.6896 : 5 : 5, stated as paid to the King of England, is thus marked, vj$^{m}$ . viij$^{c}$ . iiij$^{xx}$ . xvj . ŧj . v . s̃ . v . đ. It may be observed, that, in Scotland, the contraction ʒm for M, or *one thousand*, is still used, in the dates of charters and other legal instruments.

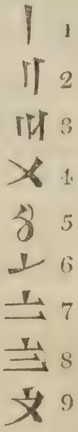
## Note III.—Page 10.

The Chinese had, from the earliest times, constructed a system of numerals, similar in many respects to what the Romans probably derived from their Pelasgic ancestors. It is only to be observed, that the Chinese mode of writing is the reverse of ours ; and that, beginning at the top of the leaf, they descend in parallel columns to the bottom, proceeding, however, from right to left, as practised by most of the Oriental nations.—Instead of the vertical lines used by the Romans, we therefore meet with horizontal ones in the Chinese notation. Thus, *one* is represented by a horizontal stroke, with a sort of barbed termination ; *two* by a pair of such strokes ; and *three* by as many parallel strokes ; the mark for *four* has four strokes, with a sort of flourish ; three horizontal strokes, with two vertical ones, form the mark for *five ;* and the other symbols exhibit the successive strokes abbreviated, as far as *nine. Ten* is figured by a horizontal stroke, crossed with a vertical score, to show that the first rank of the Binary Scale was completed ; *an hundred* is signified by two vertical scores, connected by three short horizontal lines ; *a thousand* is represented by a sort of double cross ; and the other ranks, ascending to *an hundred millions*, have the same marks successively compounded. The annexed figures are very exactly copied from the impressions given in Dr Marshman's Elements of the Chinese Grammar, a work printed with metallic types, instead of the ordinary wooden blocks, at the Baptist Missionary Press at Serampore in 1814.

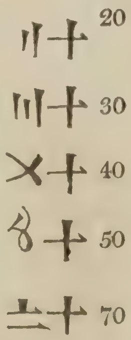
| | | | | | |
|---|---|---|---|---|---|
| 1 | 一 | *Yĭh.* | 10 | 十 | *Shih.* |
| 2 | 二 | *Irr.* | 100 | 百 | *Pŭh.* |
| 3 | 三 | *San.* | 1000 | 千 | *Ts'hyen.* |
| 4 | 四 | *Sè.* | 10,000 | 萬 | *Wàn.* |
| 5 | 五 | *Ngóo.* | 100,000 | 億 | *Eè.* |
| 6 | 六 | *Lyeù.* | 1,000,000 | 兆 | *Chaò.* |
| 7 | 七 | *Ts'hĭh.* | 10,000,000 | 京 | *King.* |
| 8 | 八 | *Păh.* | 100,000,000 | 垓 | *Kyai.* |
| 9 | 九 | *Kyéu.* | | | |

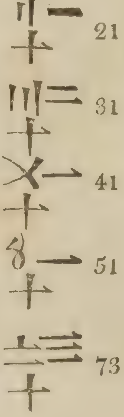
The numbers *eleven*, *twelve*, &c. are represented by putting the several marks for *one*, *two*, &c. the excesses above *ten*, immediately below its symbol. But, to denote *twenty*, *thirty*, &c. the marks of the multiples *two*, *three*, &c. are placed above the symbol for *ten*. This distinction is pursued through all the other cases. Thus, the marks for *two*, *three*, &c. placed over the symbols of *an hundred* or of *a thousand*, signify so many *hundreds* or *thousands*.—The character for *ten thousand*, called *wàn*, appears to have been the highest known at an early period of the Chinese history, since, in the popular language at present, it is equivalent to *all*. But the Greeks themselves advanced no farther. In China, *wàn*, *wàn*, signifies *ten thousand* times *ten thousand*, or *an hundred millions;* though there is also a distinct character for this very large number. In the Eastern strain of hyperbole, the phrase *wàn*, *wàn*, far out-doing *a thousand years*, the measure of Spanish loyalty, is the usual shout of Long Live the Emperor! The Chinese character *chaò* for a *million*, though not of the greatest antiquity, is yet as old as the time of Confucius. The characters for *ten*, and for an *hundred*, *millions*, are not found in their oldest books, but occur in the Imperial Dictionary.

Such is the very complete but intricate system of Chinese numerals. It has been constantly used, from the remotest times, in all the historical, moral and philosophical compositions of that singular people. The ordinary symbols for words, or rather things, are, in their writings, genealy blended with skill among those characters. But the Chinese merchants and traders have transformed this system of notation into another, which is more concise, and better adapted for the details of business. The changes made on the elementary characters, it will be seen, are not very material. The *one*, *two* and *three* are represented by perpendicular strokes ; the symbols for *four* and *five* are altered : *six* is denoted by a short score above an horizontal stroke, as if to signify that *five*, the half of the index of the scale, had been counted over ; *seven* and *eight* are expressed by the addition of one and two horizontal lines ; and the mark for *nine* is composed of that for six, or perhaps at first a variety of five, joined to that of *four*.

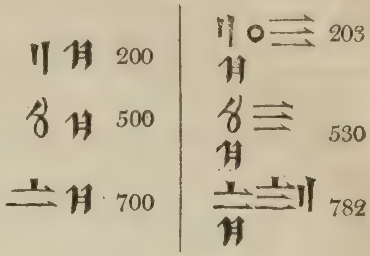



To represent *eleven*, *twelve*, &c. in this mode, a single stroke is placed on the left of the cross for ten, and the several additions of *one*, *two*, &c. annexed on the right. From *twenty* to *an hundred*, the signs of the multiples are prefixed to the mark for *ten*.

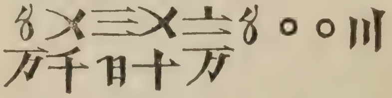


The same method is pursued through the hundreds, the marks of the several multiples being always placed on the left hand before the contracted symbol of *pŭh*, or *an hundred.* The additions are made on the right, with a small cipher or circle (0), called *ling*, when necessary, to separate the place of units. The distinction between *two hundred and three*, and five hundred and thirty, deserves to be particularly remarked.



A similar process extends to the notation of thousands; but, for ten thousand, the character *wàn* is abbreviated. As a specimen of their combination, we select the following complex expression, which denotes 543,475,003.



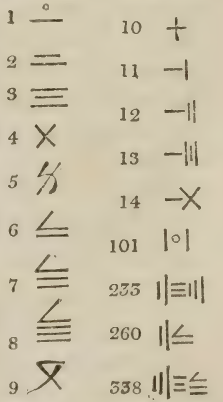
The same number would be thus represented in the regular system of Chinese notation: Where the first column on the right hand presents the marks for *fifty* and *four*, with the interjacent character *wàn*, or *ten thousand;* the next column to the left has the several marks for a *thousand*, *three*, and *an hundred;* the middle column exhibits the symbols of *forty* and of *seven;* the adjacent column repeats the character *wàn*, or *ten thousand*, and then presents those for *five* and *a thousand;* and the last column has the symbol *ling*, or the *rest*, which fills up the blank, with the mark for *three.*

五萬四千三百四十七萬五千零三

The last expression seems abundantly complicated, and yet it is unquestionably simpler and clearer than the corresponding notation with Roman numerals. From such an intricate example, the imperfection of the Roman system will appear the more striking.

The abbreviated process of the Chinese traders was probably suggested by some communication with India, where the admirable system of denary notation has, from remote ages, been understood and practised. The adoption of a small cipher to fill the void spaces, was a most material improvement on the very complex character *ling*, used formerly for the same purpose.

About the close of the seventeenth century, the Jesuit missionaries Bouvet, Gerbillon, and others, then residing at the Court of Pekin, and able mathematicians, appear to have still farther improved the numeral symbols of the Chinese traders, and reduced the whole system to a degree of simplicity and elegance of form scarcely inferior to that of our modern ciphers. With these abbreviated characters they printed, at the imperial press, Vlacq's *Tables of Logarithms*, extending to ten places of decimals, in a beautiful volume, of which a copy was presented by Father Gaubil on his return to Europe, about the year 1750, to the Royal Society of London. No more than nine characters, it will be seen, are wanted, the upright cross + for *ten* being a mere redundancy. The marks for *one*, *two*, and *three*, consist of parallel strokes as before; an oblique cross × denotes *four*; and a sort of bisected ten signifies *five*. This symbol again, being contracted into the angular mark ∠, and combined with *one*, *two*, or *three* strokes drawn below it, represents *six*, *seven*, or *eight*; and still more abridged and annexed to the sign of four, it denotes *nine*. The distinction of units, tens, hundreds, &c. is indicated by giving the strokes alternately an horizontal and vertical position; while the blanks or



vacant bars are expressed by placing small zeros.—The very important collection of logarithmic tables just mentioned, was printed by the command of Kang-shi, the second Emperor of the present dynasty, a man of enlarged views, who governed China with dignity and wisdom during a long course of years. This enlightened Prince was much devoted to the learning of Europe, and is reported to have been so fond of calculation, as to have those tables abridged and printed in a smaller character, which precious volume he carried constantly fastened to his girdle.

## Note IV.—Page 11.

The oriental nations appear generally to have represented the numbers as far as *one thousand*, by dividing their alphabet into three distinct classes. But the Hebrew, the rudest and poorest of all written languages, having only twenty-two letters, could advance no farther than 400 ; and to exhibit 500, 600, 700, 800, and 900, it had recourse to the clumsy expedient of addition, by joining 400 and 100, 400 and 200, 400 and 300, 400 and 400, and 400 with 400 and 100. The Arabic alphabet, containing twenty-eight letters, supplied fully the three classes. It is very remarkable, that, when these letters are employed to signify numbers, they are written, in the customary way, from right to left ; but in adopting the peculiar numeral character appropriately styled Indian, the order is inverted, or proceeds from the left to the right. The Arabians and Persians have also another set of symbols called the *Diwāni* to express numbers, consisting merely of disguised and contracted words. This system extends as far as 400,000, and is much used in the East for keeping of accounts.

## Note V.—Page 11, 12.

The Greeks, to represent numbers, distinguished their alphabet into three classes, in each of which they inserted a supplementary character. The first class exhibited the nine *units*, called μοναδικαι or ἑναδικαι; the next class denoted the series of *tens*, thence named δεκαδικαι; and the third class expressed the successive *hundreds*, which were termed ἑκατονταδικαι. To complete those classes, the mark ϛ, called *episêmon*, was introduced among the units after ε to denote *six*, and the *koppa* and *sanpi*, represented by Ϟ, ϟ, or ϡ, terminated respectively the range of *tens* and *hundreds*, or expressed *ninety* and *nine hundred*. The notation of numbers, as far as one thousand, was therefore effected in this way, the artificial word αιϱ, which contains the letters that commence each series, serving to aid the recollection of their order.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| α. | β. | γ. | δ. | ε. | ϛ. | ζ. | η. | θ. |
| 1. | 2. | 3. | 4. | 5. | 6. | 7. | 8. | 9. |
| ι. | κ. | λ. | μ. | ν. | ξ. | ο. | π. | ϟ. |
| 10. | 20. | 30. | 40. | 50. | 60. | 70. | 80. | 90. |
| ϱ. | σ. | τ. | υ. | φ. | χ. | ψ. | ω. | ϡ. |
| 100. | 200. | 300. | 400. | 500. | 600. | 700. | 800. | 900. |

But the same letters, by having an *iôta* subscribed, were augmented *one thousand times*. Thus, $\underset{,}{\alpha}$, $\underset{,}{\beta}$, $\underset{,}{\gamma}$ &c. denoted 1000, 2000, 3000, &c. or the χιλιαδικαι; $\underset{,}{\iota}$, $\underset{,}{\kappa}$, $\underset{,}{\lambda}$, &c. expressed 10,000, 20,000, 30,000, &c. or the δεκαδαι χιλιαδικαι; and $\underset{,}{\varrho}$, $\underset{,}{\sigma}$, $\underset{,}{\tau}$, &c. represented 100,000, 200,000, 300,000, &c. The values of the several characters could likewise be augmented *ten thousandtimes*, by placing under them the initial letter M of the word Μυϱια. Thus, $\underset{\mathrm{M}}{\varrho}$ signified *one million*. Another mode of representing large numbers was to superscribe repeated dots. Thus $\ddot{\alpha}$ expressed *ten thousand*, or the commencement of the series of *myriads*, or μυϱιονταδικαι ἁπλαι; and $\overset{::}{\alpha}$ denoted *one hundred millions*, beginning the μυϱιονταδικαι διπλαι, or the successive squares of the former set. As an example, the number 3,280,196,529 would have been written by the Greeks, $\overset{::}{\lambda}.\ \overset{::}{\beta}.\ \ddot{\eta}.\ \ddot{\iota}.\ \ddot{\theta}.\ \varsigma.\ \varphi.\ \kappa.\ \theta$, and read μυϱιαδικῶν διπλῶν λβ. μυϱιαδικῶν ἁπλῶν ηιθ. και ἑξακισχιλια πεντακοσια εικοσιν εννεα.

Such is the beautiful system of Greek numerals, so vastly superior in clearness and simplicity to the Roman combination of strokes. It was even tolerably fitted as an instrument of calculation. Hence the Greeks early laid aside the use of the *abacus;* while the Romans, who never showed any taste for science, were, from the total inaptitude of their numerical symbols, obliged at all times to practise the same laborious manipulation.

Allusions to the Grecian mode of notation occur frequently in the ancient classics. Hence the point of the following epigram:

> Ἑξ ὥραι μόχθοις ἱκανώταται. αἱ δὲ μετ' αὐτὰς
> Γράμμασι δεικνύμεναι ΖΗΘΙ λέγουσι βροτοῖς.

The meaning of the passage is, that the space from sunrise to noon, or the seventh hour, which was consequently denoted by the letter *zêta*, being consumed in labour, the rest of the day may be fairly devoted to relaxation.

## Note VI.—Page 97.

In the early ages of the Republic, the private Romans were accustomed to register their time, by casting every day a *lapillus*, or little pebble into an urn. If the day closed happily, a white pebble was chosen; but if they deemed it unfortunate, they selected a black one.

> Hunc, Macrine, diem numera *meliore lapillo*,
> Qui tibi labentes apponit candidus annos. Pers. *Sat.* II. 1, 2.

The *Abacus*, or *Tabula Logistica*, with its furniture, is frequently mentioned in the Classics.

> Quo pueri magnis e centurionibus orti,
> Lævo suspensi *loculos tabulamque* lacerto. Hor. *Sat.* I. vi. 73.

For the purpose of elementary education, this table or board was strewed with sand.

> Nec qui *abaco* numeros, et secto in pulvere metas
> Scit risisse vafer. Pers. *Sat.* I. 132.

The sand used was, according to Martianus Capella, of a sea-green colour:

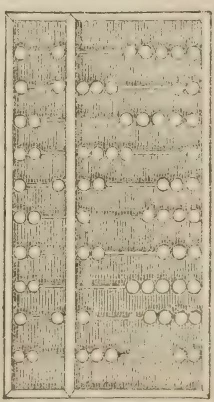
Sic *abacum* perstare jubet, sic *tegmine glauco*
Pandere pulvereum formarum ductibus æquor.
Lib. vii. De *Arithmetica.*

The Abacus appears to have continued in familiar use among the modern nations of Europe, even till a recent period. The counters or pebbles were, from the corruption of the word *algorithm,* called in England *augrim,* or *awgrym,* *stones.* Thus, in Chaucer's lively description of the chamber of Clerk Nicholas:

His almageste and bokes grete and smale,
His astrelabre, longing for his art,
His augrim stones, layen faire apart
On shelves couched at his beddes head.
*Miller's Tale,* v. 22—24.

## Note VII.—Page 97.

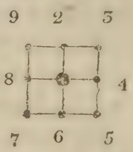
It is observed in the text, that the *Computing Table,* or *Swan-Pan* of the Chinese, nearly resembles the Roman Abacus. It consists of a small oblong board surrounded by a high ledge, and parted downwards near the left side by a similar ledge. It is then divided horizontally by ten smooth and slender rods of bamboo, on which are strung two small balls of ivory or bone in the narrow compartment, and five such balls in the wider compartment; each of the latter on the several bars denoting *one,* and each of the former, for the sake of expedition, expressing *five.* The progressive bars, descending after the Chinese manner of writing, have their values increased tenfold at each step. The arrangement here figured will hence signify, reckoning downwards, 5,804,712,063. The Swan-Pan advances the length of ten billions, and therefore a thousand times farther than the Roman Abacus. But the capital im-



provement which the Chinese had made, was, by commencing the units from any particular bar, to represent the decimal subdivision on the same instrument. Yet this most useful extension of the denary scale, however obvious it may now appear, was unknown in Europe before the time of Stevinus.

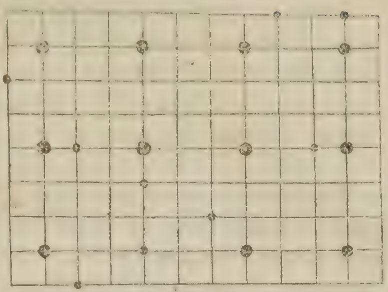
## Note VIII.—Page 97.

The famous Dr Saunderson, professor of mathematics in the University of Cambridge, who had the misfortune to be totally deprived of sight by small-pox when only twelve months old, contrived a very ingenious kind of *Abacus*, well adapted to his forlorn case, and requiring comparatively few implements for its operations. It consisted of a smooth thin board, rather more than a foot square, divided by equidistant vertical and horizontal lines about the tenth part of an inch apart, and having their intersections perforated with holes. Conceiving the surface to be crossed by belts composed of double lines, and consequently distinguished into a multitude of quartered squares, he assigned a digit to each of these clusters, by planting a large headed pin in the central hole, and combining with it another small headed one, stuck into the various holes round the margin. The large pin placed alone in the centre of the compound square expressed merely the 0, or zero; and the small pin being substituted, signified 1. The rest of the procedure was quite regular. On replacing the large pin, a small one stuck in the hole directly above it denoted 2; in the hole on the right corner, 3; in the hole below this, 4; in the next corner, 5;—thus, following the exterior circuit, with 7 and 8, till 9 is signified by placing the variable pin in the upper and left corner.




In other respects, the board had precisely the same construction as the *Abacus*. The position of each central pin determined the value of its digit in reference to *units*, *tens*, *hundreds*, *thousands*, &c. But the better to distinguish those belts or double bars, the spaces were marked by notches along the outer edge of the board. A large stock of pins with their points cut off, was kept ready for use in two separate boxes.

This curious expedient for calculation will be clearly understood from the inspection of the annexed diagram, in which the numbers 7032,4608, and 5190, are denoted, running from left to right, and reckoning from the top.



Dr Saunderson is said to have acquired, from assiduous practice, great facility and quickness in performing arithmetical operations.

## Note IX.—Page 100, 101.

The mode of signifying numbers by different inflexions of the fingers, is of very great antiquity. The symbols for *one* and for *seven* are nearly the same, only the little finger is extended a joint farther in the latter. *Three* and *eight* are distinguished likewise, by the different extension of the two last fingers of the hand. Similar differences may be perceived in the expressions of other numbers.

Those digital symbols were quite familiar to the ancients. According to Pliny, the image of Janus, or the Sun, was formerly moulded with the fingers bent, to signify the 365 days of the solar year. *Janus geminus à Numâ rege dicatus, qui pacis bellique argumento colitur, digitis ita figuratis, ut trecentorum sexaginta quinque dierum nota per significationem anni temporis, et ævi se Deum indicaret.* Hist. Nat. Lib. xxxiv. 7.

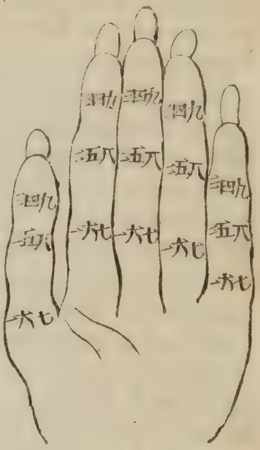
In allusion to the circumstance that *hundreds* came to be expressed on the right hand, many passages occur in the Classics. *In digitos rem redire, in digitos mittere, in digitis constituere,*—are expressions which occur in Cicero. Juvenal thus describes the aged Nestor:

> Rex Pylius (magno si quicquam credis Homero)
> Exemplum vitæ fuit à cornice secundæ.
> Felix nimirum, qui tot per secula mortem
> Distulit, atque suos jam *dextra* computat annos.
>
> *Sat.* x. 246

Some commentators would even explain, from the same practice of numeration, the allegorical description of Wisdom in the Proverbs of Solomon :

Length of days is in her *right hand*, and in her *left hand* riches and honour. Prov. iii. 16.

The Chinese have also contrived a very neat and simple kind of digital signs for denoting numbers, greatly superior, both in precision and extent, to the method practised by the Romans. Since every finger has three joints, let the thumb-nail of the other hand touch those joints in succession, passing up the one side of the finger, down the middle, and again up the other side, and it will give *nine* different marks, applicable to the *Denary Scale* of arrangement. On the little finger, those marks signify *units*, on the next finger *tens*, on the mid-finger *hundreds*, on the index *thousands*, and on the thumb *hundred thousands*. With the combined positions of the joints of the one hand, therefore, it was easy to advance by signs as far as a *million*. To illustrate more fully this ingenious practice, I have here copied, from a Chinese elementary treatise of education, the figure of a hand, noted at the several joints of each finger, by characters along the inside, corresponding to 1, 2, and 3;



down the middle, by those answering to 4, 5, and 6; and again up the outside, by characters expressing 7, 8, and 9. The length of the projecting nails betokens it to be the hand of one of the literary or higher classes in society. The merchants of China are accustomed, it is said, to conclude bargains with each other by help of those signs; and often, prompted by selfish or fraudulent views, conceal the pantomime from the knowledge of bystanders, by only seeming to seize the hand with a hearty grasp.

Note X.—Pages 106—110.

I am sorry, after all the pains I have taken, at not being able to come to a more definite conclusion relative to the origin of our present numeral characters. There is strong ground to suspect that the Hindus obtained the knowledge of those figures from Upper Asia or perhaps Tartary. The humble attainments of this people are not entitled to claim any very high antiquity. The expedition of Alexander the Great into the East, opened a channel of learned intercourse; but, according to the testimony of his Generals, Megasthenes and Nearchus, the people of India were at that period entirely ignorant of letters, nor had they acquired any skill in arithmetic about the time of Arrian and Philostratus. The Sanscrit alphabet is asserted by Anquetil to have been formerly distributed into three classes, which, as in other languages, were employed to denote the successive ranks of units, tens, and hundreds; and Croze contends, that the followers of Zoroaster on the Malabar coast continued long afterwards to represent numbers by means of letters. This application of the alphabet has no doubt, for some ages, given place among the Hindus to the simpler and more perfect system of digital characters. At what epoch such a mighty change was effected, it would be difficult to conjecture. The most ancient monument of the Sanscrit numerals at present believed to exist, is a Royal Grant of land engraved on a large folding plate of copper discovered in the ruins of Mongueer, which has been described in the first volume of the Asiatic Researches by the learned Dr Wilkins; and its date, the 33d of *Sombot*, referred by him to the 23d year before the Christian æra. This curious document may be genuine; but the Brahmins of India, like their brethren the monks of Europe, are well known to have been strongly addicted to the pious fraud of forging charters and other deeds favourable to the interests of their order.

The oldest treatise on arithmetic possessed by the Hindus, the *Lilavati*, remounts no higher than the eleventh century of our æra. This famous composition, to which the vanity and ignorance of that people claim a divine original, is but a very poor performance, containing merely a few scanty precepts couched in obscure memorial verses. The examples annexed to those rules, often written probably by later hands in the margin, are generally trifling and ill-chosen. Indeed, the *Lilavati* exhibits nothing that deserves the slightest notice, except the additions made by its Persian commentators. The Hindus had not the sagacity to perceive the various advantages to be derived from the denary notation. They remained entirely ignorant of the use of decimal fractions, with which their acute neighbours, the Chinese, have been familiarly acquainted from the remotest ages. Their numerical operations are unnecessarily complicated, following closely the procedure which the application of an alphabet had obliged the Greeks to employ. It would seem that, while the Hindus communicated their numerals to the Arabians, they were glad, in return, to accept the mode of calculation adapted to a very different and inferior system of notation.

## Note XI.—Page 111.

I strongly suspect that the person styled Leonard of Pisa or Fibonacci, lived two hundred years later than the period assigned in the text. Fabricius places him at the beginning of the fifteenth century. But Cossali, in a ponderous work, entitled, *Origine, trasporto in Italia, è primi progressi in essa dell' Algebra,* and printed at Parma in 1797, maintains his antiquity in a triumphant tone, and discourses with more than Italian prolixity on his merits. Still, however, the claims urged for Fibonacci to such an early date, appear to rest on very slender authority. Targioni Tozzetti, about sixty years ago, discovered, in the celebrated Magliabecchi library, a manuscript with this inscription: *Incipit Liber Abbaci, compositus a Leonardo, filio Bonacci, Pisano, in anno* 1202. Some time afterwards, Zaccaria found, in the Ambrosian library, another enlarged manuscript by the same author, treating of mensuration or geodesia, and bearing date 1220. Now, we are left merely to guess at the age of these copies, nor can the dates attached to them be considered as fixing any thing but the opinion of the transcribers. Besides, in the older forms of the digits, the character of 4 very nearly resembled that of 2. I am inclined, therefore, to believe, that both those tracts of the son of Bonacci were only the same work, and ought to be referred to the year 1420. This view of the matter would reconcile the various discordant facts. All the early treatises of Arithmetic in Europe, being formed after Arabian models, had commonly subjoined to them some portions of Algebra and of Practical Geometry. But is it not in the highest degree improbable that, if Leonard had once taught his townsmen of Pisa the use of the denary numerals, an art so useful and so simple could have been ever lost? No certain traces of the digital arithmetic are found among the Christians, before the close of the fourteenth century; and from this epoch, the knowledge of it, diffused by commercial intercourse, appears to have been soon conveyed from Italy to France, Germany and England. Treatises on the subject of calculation were now composed in different parts of Europe, and the noble art of printing came most opportunely to multiply their circulation and perpetuate their influence.

## Note XII.—Page 127.

Pythagoras brought from the East a passion for the mystical properties of numbers, under the veil of which he probably concealed some of his secret or esoteric doctrines. He regarded *Numbers* as of divine origin, the fountain of existence, and the model and archetype of all things. He divided them into a variety of different classes, to each of which were assigned distinct properties. They were prime or composite, perfect or imperfect, redundant or deficient, plane or solid; they were triangular, square, cubic, or pyramidal. *Even* numbers were held by that visionary philosopher as feminine, and allied to earth; but the *odd* numbers were considered by him as endued with masculine virtue, and partaking of the celestial nature.

*Unit*, or *monad*, was held as the most eminently sacred, as the parent of scientific numbers. *Two*, or the *duad*, was viewed as the associate of the *monad*, and the mother of the elements, and the recipient of all things material; and *three*, or the *triad*, was regarded as perfect, being the first of the masculine numbers, comprehending the beginning, middle, and end, and hence fitted to regulate by its combinations the repetition of prayers and libations. It was the source of love and symphony, the fountain of energy and intelligence, the director of music, geometry, and astronomy. As the *monad* represented the Divinity, or the Creative Power, so the *duad* was the image of Matter; and the *triad*, resulting from their mutual conjunction, became the emblem of Ideal Forms.

But *four*, or the *tetrad*, was the number which Pythagoras affected to venerate the most. It is a square, and contains within itself all the musical proportions, and exhibits by summation all the digits as far as ten, the root of the universal scale of numeration; it marks the seasons, the elements, and the successive ages of man; and it likewise represents the cardinal virtues, and the opposite vices. The ancient division of mathematical science into Arithmetic, Geometry, Astronomy, and Music, was four-fold, and the course was therefore termed a *tetractys*, or *quaternion*. Hence Dr Barrow would explain the oath familiar to the disciples of Pythagoras: "I swear by him who communicated the *Tetractys*."

*Five*, or the *pentad*, being composed of the first male and female numbers, was styled the number of the world. Repeated any how by an odd multiple, it always re-appeared; and it marked the animal senses, and the zones of the globe.

*Six*, or the *hexad*, being composed of its several factors, was reckoned perfect and analogical. It was likewise valued, as indicating the sides of the cube, and as entering into the composition of other important numbers. It was deemed harmonious, kind and nuptial. The third power of 6, or 216, was conceived to indicate the number of years that constitute the period of metempsychosis.

*Seven*, or the *heptad*, formed from the junction of the *triad* with the *tetrad*, has been celebrated in every age. Being unproductive, it was dedicated to the virgin Minerva, though possessed of a masculine character. It marked the series of the lunar phases, the number of the planets, and seemed to modify and pervade all nature. It was called the horn of Amalthæa, and reckoned the guardian and director of the universe.

*Eight*, or the *octad*, being the first cube that occurred, was dedicated to Cybelé, the mother of the Gods, whose image in the remotest times was only a cubical block of stone. From its evenly composition, it was termed Justice, and made to signify the highest or inerratic sphere.

*Nine*, or the *ennead*, was esteemed as the square of the *triad*. It denotes the number of the Muses, and, being the last of the series of digits, and terminating the tones of music, it was inscribed to Mars. Sometimes it received the appellation of Horizon, because, like the spreading ocean, it seemed to flow round the other numbers within the Decad: For the same reason, it was also called Terpsichore, enlivening the productive principles in the circle of the dance.

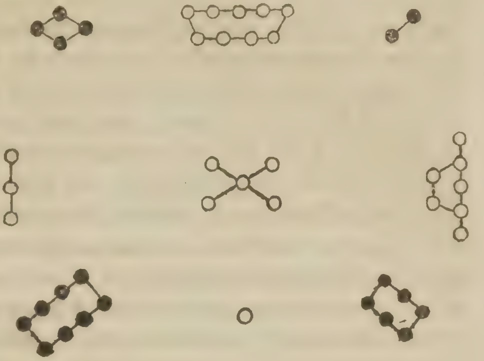
*Ten*, or the *decad*, from the important office which it performs in numeration, was, however, the most celebrated for its properties. Having completed the cycle, and begun a new series of numbers, it was aptly styled *apocatastasic* or periodic, and therefore dedicated to the double-faced Janus. It had likewise the epithet of Atlas, the unwearied supporter of the world.

The cube of the *triad*, or the number *twenty-seven*, expressing the time of the moon's periodic revolution, was supposed to signify the power of the lunar circle. The quaternion of celestial numbers, *one*, *three*, *five*, and *seven*, joined to that of the terrestrial numbers, *two*, *four*, *six*, and *eight*, compose the number *thirty-six*, the square of the first perfect number *six*, and the symbol of the universe, distinguished by wonderful properties.

But it would be endless to recount all the visions of the Pythagorean school; nor should we stop to notice such fancies, if, by a perpetual descent, the dreams of ancient Philosophers had not, in the actual state of society, still tinctured our language, and mingled themselves with the various institutions of civil life. The mystical properties of numbers, originally nursed in the sombre imagination of the Egyptians, were eagerly embraced by the Jewish Cabbalistic writers, and afterwards implicitly adopted by the Fathers of the Christian Church. But those fancies maintained an ascendancy in public belief until a very late period, nor were the Reformers themselves exempt from their influence. Luther, whose vigorous mind was yet deeply tinctured with the credulity of his age, was accustomed to venerate certain numbers with a species of idolatry. Peter Bungus, canon of Bergamot, published, in 1585, a thick quarto, *De mysticis numerorum significationibus*, chiefly with a view to explain some passages in the Old and New Testament. The famous number of the Beast, 666, which has so often tortured the ingenuity of the expounders of the Apocalypse, is regarded by some Divines as of Egyptian descent, the archetype of the three monads, and combining the genial and siderial powers; being indeed only the sum of all the terms of the magic square of 6, the first of the perfect numbers, and dedicated to the Sun. But we still see the predilection for Luther's favourite number, *seven*, strongly marked in the customary term of apprenticeships, in the period acquired for obtaining academical degrees, and in the legal age of majority.

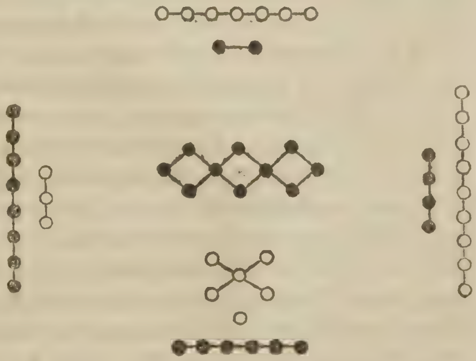
## Note XIII.—Page 128.

The Chinese appear, from the remotest epochs of their empire, to have entertained the same admiration of the mystical properties of numbers that Pythagoras imported from the East. Distinguishing numbers into even and odd, they considered the former as terrestrial, and partaking of the feminine principle *Yang*; while they regarded the latter as of celestial extraction, and endued with the masculine principle *Y*. The even numbers were represented by small black circles, and the odd ones by similar white circles, variously disposed and connected by straight lines. The sum of the five even numbers, *two*, *four*, *six*, *eight*, and *ten*, being *thirty*, was called the number of the *Earth*; but the sum of the five odd numbers *one*, *three*, *five*, *seven*, and *nine*, or *twenty-five*, being the square of *five*, was styled the number of *Heaven*. The nine digits were likewise grouped in two different ways, termed the *Lo-chou*, and the *Ho-tou*. The former expression signifies the *Book of the river Lo*, or what the Great Yu saw delineated on the back of the mysterious tortoise which rose out of that river: It is here represented.



*Nine* was reckoned the head, and *one* the tail of the tortoise; *three* and *seven* were considered as its left and right shoulders, and *four*, and *two*, *eight* and *six*, were viewed as the fore and the hind feet. The number *five*, which represented the heart, was also the emblem of Heaven. It need scarcely be observed, that this group of numbers is nothing but the common *magic-square* of the *nine* digits, each row of which makes up *fifteen*.

As the *Lo-chou* had the figure of a square, so the *Ho-tou* had that of a cross. It is what the Emperor Fou-hi observed on the body of the horse-dragon, which he saw spring out of the river Ho. The central number was *ten*, which, it is remarked by the commentators, terminates all the operations on numbers.



## Note XIV.—Page 132.

Note XIV.—Page 132.

From the data given by Ptolemy, the ratio of the diameter to the circumference of a circle may be derived somewhat differently. Since the length of the arc of one degree, considered as equal to its chord, is 1.2′.50″ in sexagesimal parts of the radius; the circumference of a circle, whose diameter is unit, will be denoted by $3.8'\frac{1}{2}$, the triple of that measure. Wherefore, the diameter is to the circumference of a circle, as 1 to $3\frac{8\frac{1}{2}}{60}$, or as 60 to $188\frac{1}{2}$, that is, as 120 to 377. If, from these numbers again, the corresponding terms 7 and 22 of the Archimedian approximation be respectively subtracted, there will remain the ratio of 113 to 355, which is the expression assigned by Metius.

It has been very generally supposed, that no nearer approximation to the ratio of the circumference to the diameter of a circle, than that of Archimedes, was known to the Mathematicians of Europe before the time of Vieta. This opinion, however, is unquestionably erroneous. Purbach, the great restorer of science, who flourished more than one hundred years before that period, expressly says, in his introduction to a compend of Ptolemy's Almagest, which he had been able to study only through the medium of a corrupt Latin version from the Arabic, that the Greek astronomers made the circumference of a circle to consist of 377 degrees, of which the diameter contains 150 (120), having deduced this estimate from the side of the inscribed decagon, which amounts to 27 (37) degrees and 4 minutes nearly. But no such passage occurs in Ptolemy's preliminary book on chords. It is true, that from the side of the inscribed decagon, or 37°4′55″, he computed the chord of 24°, or the difference between 36° and 60°, and afterwards derived the chords of the successive bisections of this arc, till he found that of $1\frac{1}{2}$° to be 1°34′15″, which gives, consequently, in the same proportion, 1°2′50″ for the chord of 1°. The Arabian commentators must, therefore, have spontaneously drawn the inference already stated, that the diameter of a circle, being divided into 120 parts, will include 377. They had also given a different form to the same conclusion ; for Purbach subjoins, that some authors make the diameter to be to the circumference in the ratio of 20000 to 62832, the same evidently as 1 to the decimal expression 3.1416. He further adds, that the Indians allege the ratio to be the same as that of 1 to the square root of 10, if such a number were capable of extraction. This differs widely, however, from the truth, though the famous philologer Joseph Scaliger afterwards proposed it, with most presumptive dogmatism, as an absolute quadrature of the circle.

It may be remarked, that the sines near the beginning of the quadrant, calculated to seven places of figures by Purbach's disciple and successor, Regiomontanus, would afford a nearer approximation. Thus, the sine of half a degree expressed sexagesimally and centesimally, is .52359, which, being multiplied by 6, and marked decimally, gives 3.14154, for the circumference of a circle whose diameter is 1.

## Note XV.—Page 174.

It should be remarked, that the ancient Greeks distinguished the Theory from the Practice of Arithmetic, by separate names. The term *Arithmetic* itself was restricted by them to the science which treats of the nature and general properties of numbers ; while the appellation *Logistic* was appropriated to the collection of rules framed to direct and facilitate the common operations of calculation. The ancient systems of Arithmetic, accordingly, from the books of Euclid to the numerical treatise of Boëthius, are merely speculative, and often abound with fanciful analogies.

The Logistic or Practical Arithmetic of the Greeks, though unavoidably encumbered by the triplicate form of their numeral notation, had, by successive improvements, attained to a remarkable degree of perfection. A few select examples will explain the mode of operation. Suppose the number 862 were to be multiplied by 523, the process would have been thus conducted by the Greek Arithmeticians, the several steps being, for the sake of clearness, marked likewise in modern numerals.

Here, in the *first* range, $\varphi$ multiplied into $\omega$, being the same as 40, the product of 8 and 5 augmented ten thousand times, is consequently denoted by $\ddot{\mu}$ or $\underset{\text{M}}{\mu}$; $\phi$ multiplied into $\xi$ gives the same result as 30 or 5 times 6 increased a thousand fold, and therefore expressed by $\ddot{\gamma}$ or $\underset{\text{M}}{\gamma}$; and $\varphi$ multiplied into $\beta$, evidently makes a thousand or $\underset{,}{\alpha}$. In the *second* range

| | |
|---|---|
| $\omega\ \xi\ \beta$ | 862 |
| $\phi\ \varkappa\ \gamma$ | 523 |
| $\ddot{\mu}\ \ddot{\gamma}\ \underset{,}{\alpha}$ | 40 . . . . |
| | 3 . . . . |
| $\alpha\ \varsigma$ | 1 . . . |
| | 16 . . . |
| $\underset{,}{\alpha}\ \sigma\ \mu$ | 12 . . |
| | 4 . |
| $\beta\ \upsilon$ | 24 . . |
| | 1 8 0 |
| $\varrho\ \pi\ \varsigma$ | 6 |
| $\ddot{\mu}\ \ddot{\varepsilon}\ \omega\ \varkappa\ \varsigma$ | 4 5 0 8 2 6 |

$\dot{\varkappa}$ multiplied into $\omega$ gives the same product as 8 repeated twice, and then augmented a thousand times, or denoted by $\ddot{\alpha}\ \underset{,}{\varsigma}$; $\varkappa$ multiplied to $\xi$ is equivalent to 6 repeated twice, and afterwards increased an hundred fold, or expressed by $\underset{,}{\alpha}\ \sigma$; and $\varkappa$ multiplied by $\beta$ gives 40, the value of $\mu$. In the third range, $\gamma$ multiplied into $\omega$ produces 2400, which is denoted by $\underset{,}{\beta}\ \upsilon$; $\gamma$ multiplied into $\xi$ makes 180 or $\varrho\ \pi$; and, lastly, $\gamma$ multiplied into $\beta$ gives $\varsigma$, the symbol of 6. Collecting now the scattered members into one sum, the result of the multiplication of $\omega\,\xi\,\beta$ by $\phi\,\varkappa\,\gamma$ is $\ddot{\mu}\ \ddot{\varepsilon}\ \omega\ \varkappa\ \varsigma$, or 450, 826.

But the notation of the Greeks was not at all adapted to the descending scale. They had no decimals, and to express vulgar fractions they took two different methods. 1. If the numerator happened to be unit, the denominator was indicated by an accent. Thus, $\delta'$ signified $\frac{1}{4}$, and $\varkappa\varepsilon'$ $\frac{1}{25}$; but *one-half*, being of more frequent occurrence, was denoted by a particular character, varying in its form, $C$, $\angle$, $C'$, or Ϛ. 2. In other cases, it was the practice of the Greeks to write the denominator, as we do an exponent, a little above the numerator, and towards the right hand: Thus, $\beta^{\iota\alpha}$ intimated $\frac{2}{11}$, and $\pi\alpha^{\rho\varkappa\alpha}$ $\frac{81}{121}$.

In illustration of the management of fractions, I shall take an example which is rather complicated, from the commentary which Eutocius of Ascalon wrote, about the third century of our æra, on the Tract of Archimedes concerning the quadrature of the circle. Let it be required to find the square of $\underset{,}{\alpha}\omega\lambda\eta\ \theta^{\iota\alpha}$, or $1838\frac{9}{11}$.

$$\underset{,}{\alpha}\ \omega\ \lambda\ \eta\quad \theta^{\iota\alpha}$$
$$\underline{\underset{,}{\alpha}\ \omega\ \lambda\ \eta\quad \theta^{\iota\alpha}}$$
$$\underset{M}{\varrho}\ \underset{M}{\pi}\ \underset{M}{\gamma}\ \underset{,}{\eta}\ \underset{,}{\omega}\ \varsigma\ \eta\quad \beta^{\iota\alpha}$$
$$\pi\ \underset{M}{\xi\delta}\ \underset{M}{\beta}\ \underset{,}{\delta}\ \underset{,}{\varsigma}\ \upsilon\ \chi\ \nu\ \delta\quad \varsigma^{\iota\alpha}$$
$$\underset{M}{\gamma}\ \underset{M}{\beta}\ \underset{,}{\delta}\ \sampi\ \sigma\ \mu\ \kappa\ \delta\quad \varsigma^{\iota\alpha}$$
$$\underset{,}{\eta}\ \underset{,}{\varsigma}\ \upsilon\ \sigma\ \mu\ \xi\ \delta\ \varsigma\quad \varsigma^{\iota\alpha}$$
$$\omega\ \iota\ \eta\quad \beta^{\iota\alpha}$$
$$\chi\ \nu\ \delta\quad \varsigma^{\iota\alpha}$$
$$\kappa\ \delta\quad \varsigma^{\iota\alpha}$$
$$\varsigma\quad \varsigma^{\iota\alpha}$$
$$\pi\alpha^{\rho\kappa\alpha}$$
$$\tau\ \underset{M}{\lambda}\ \eta\quad \underset{,}{\alpha}\ \sigma\ \nu\ \alpha\quad \zeta^{\iota\alpha}\ \pi\alpha^{\rho\kappa\alpha}$$

or, $\tau\ \underset{M}{\lambda}\ \eta\quad \underset{,}{\alpha}\ \sigma\ \nu\ \beta\quad \lambda\zeta^{\rho\kappa\alpha}$

$1838\frac{9}{11}$
$\underline{1838\frac{9}{11}}$
1 . . . . . .
8 . . . . .
3 . . . .
8 . . .
$818\frac{2}{11}$
8 . . . . .
64 . . . .
24 . . .
64 . .
$654\frac{6}{11}$
3 . . . .
24 . . .
9 . .
24 .
$24\frac{6}{11}$
8 . . .
64 . .
24 .
64
$6\frac{6}{11}$
$818\frac{2}{11}$
$654\frac{6}{11}$
$24\frac{6}{11}$
$6\frac{6}{11}\ \frac{81}{121}$
$3381251\frac{7}{11}\ \frac{81}{121}$

or, $3,381,252\frac{37}{121}$

This complex process needs no explication: It is only to be observed, that to multiply the several integers by the fraction $\frac{9}{11}$, is the same thing as to multiply them first by 9, and then divide the product by 11.

It may be proper, likewise, to give an example of the multiplication of sexagesimals. For this purpose, I shall borrow a question proposed by Theon, to find the square of the side of a regular decagon inscribed in a circle, or the chord of 36°, which, according to Ptolemy's computation, measured in sexagesimal parts of the radius 37° 4′ 55″. The multiplication is thus effected:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| λζ | δ | νε | | | 37° | 4′ | 55″ | | |
| λζ | δ | νε | | | 37 | 4 | 55 | | |
| ͵ατξθ | ρμη | ͵βλε | | | 1369′ | 148″ | 2035‴ | | |
| | ρμη | ις | σκ | | | 148 | 16 | $220^{iv}$ | |
| | | ͵βλε | σκ | | | | 2035 | 220 | |
| | | | | ͵γκε | | | | | $3025^{v}$ |
| ͵ατοε | δ | ιδ | ι | κε | 1375′ | 4″ | 14‴ | $10^{iv}$ | $25^{v}$ |

The square now found is that of the greater segment of the radius divided into extreme and mean ratio, and consequently the same as the rectangle under the radius and the smaller segment, or 22° 55′ 5″. But the product of 60° into 22° 55′ 5″ is 1375′, differing only by a very minute defect from the actual result.

From these operations, it is not difficult to conceive how the Greeks would proceed in other cases, such as Division and the Extraction of the Square Root.

## Note XVI.—Page 175.

From the Greek word ψηφος, a *pebble*, came the verb ψηφιζειν, *to calculate*, and likewise the substantive noun ψηφοφορια, denoting *calculation*. Corresponding to ψηφος, the Romans had *calculus, a little chalk stone*, or counter. Hence, also, various phrases used in the Classics: *Hic calculus accedat—ponere calculum—calculum detrahere—decedere calculum—plurium calculis vincitur—calculos movere.* The following sentences are extracted from Cicero: " Quare nunc saltem *ad illos calculos revertamur*, quos tum abjecimus; ut non solum gloriosis consiliis utamur, sed etiam paulo salubrioribus."—" Hoc quidem est nimis exigue et exiliter *ad calculos vocare* amicitiam, ut par sit ratio datorum et acceptorum."

The verb *calculare*, and the substantive noun *calculatio*, formed by derivation long afterwards, are considered as barbarous Latinity, though they have been retained in most of the modern languages.

## Note XVII.—Page 184.

When ciphers were first introduced into Europe, it was deemed necessary to prefix a short abstract of their nature and application. These brief notices are often met with attached to old vellum almanacs, or inserted in the blank leaves of missals, and frequently intermixed among famous prophecies, most direful prodigies, and infallible remedies for scalds and burns. In such strange company, the denary characters copied in page 126 were found, but followed by a neat explication of their use.

After the present numerals had been generally adopted, it was the practice throughout Europe, to reduce the rules of Arithmetic, like those of the Latin Grammar, to memorial verses. A small tract composed on that plan, in the reign of Edward VI. by Buckley of Litchfield, a fellow of the University of Cambridge, appears at one period to have gained possession of the schools and colleges of England. It bore this title,—ARITHMETICA MEMORATIVA, *sive* COMPENDIARIA ARITHMETICÆ TRACTATIO, *non solum tyronibus, sed etiam veteranis, et bene exercitatis in ea arte viris, memoria juvandae gratia, admodum necessaria: Authore Gulielmo Buclæo, Cantabrigiensi.*

I shall here extract a few specimens.

---

DE NUMERATIONE.

——————— Numerorum signa decem sunt,
Quorum significant aliquid per se omnia, praeter
Postremum, nihili quae dicitur esse figura.
Circulus haec alias, alias quoque cyphra vocatur,
Quæ supplere locum nata est non significare.
Hi characteres prima si sede locentur,
Significant se simpliciter, positique secunda,

Significant decies se, quod si tertius illis
Obtigerit locus, ad centum se porrigit usque
Summa, locus quartus solus tibi millia fundit,
Et quartum quintus decies complectitur, huncque
Tantundem sextus superat quid multa sequens cum
Quisque locus, soleat decies âugere priorem.

*Ratio numeros tum scribendi, tum exprimendi.*

Scripturis numerum a dextris fac incipias, hinc
In lævam tendens, donec conscripseris omnes.
Post signa minimis loca quarternaria punctis.
Punctaque quot fuerint, totidem tibi millia monstrant.
A læva vero numerorum expressio fiat.

---

### DE MULTIPLICATIONE.

Est numerum in numerum diducere, multiplicare
Et cujus ductu numerus producitur, in se
Contineat toties numerum qui multiplicatum,
Multiplicans quoties in se complectitur unum.
Scribatur primo numerus, qui multiplicari
Debeat, et recte sub eodem multiplicantem
Ponito, ducatur solito mox linea more.
Et numerum primum seriei multiplicantis,
Multiplica in cunctos seriei multiplicandæ,
Inferius scribens quicquid producitur, atque
Si plures fuerint numeri tibi multiplicantes,
Omnes in numerum deducito multiplicandum,
Semper subscribens quicquid producitur, idque
Recte sub numero scribatur multiplicante.

Et quia quot fuerint numeri tibi multiplicantes,
Productos totidem numeros quoque adessse necesse est:
Idcirco hos omnes conjunge per additionem.
Subscriptus numerus, productus jure vocetur,
Nam, quam quærebas, solet hic producere summam.

*Examen.*

Divide productum, numerum per multiplicantem.
Si nihil errasti, prodibit multiplicandus.

---

### DE DIVISIONE.

Ostendit numeri quasvis divisio partes.
Ponatur numerus, suprema parte secandus,
Lineolasque duas ille supponito rectas,
Divisor sub iis ponatur parte sinistra.
Deinde vide quoties divisor contineatur

In supraposito numero quotiensque locetur
In spatio, mox et divisorem per eundem
Multiplica, totumque quod hinc provenerit, aufer
Supremo ex numero, supra ponendo relictum,
Transfigens numerum de quo subtractio facta est.
Si plures numeros contingat adesse secandos,
Divisor dextram versus tibi promoveatur.
Unam per seriem, Rursus quoque quærere oportet
Divisor quoties in eo, qui dividitur sit,
Et quotientem intra spatium deponere ut ante.
Sic reliqua adsolvis prorsus, quæcunque supersunt.
Nec labor hic quicquam distat, variatve priori.
Sin, qui dividitur, fuerit minor inferiori,
Supremo intacto, divisor progrediatur,
Et medio in spatio ponatur cyphra, modoque,
Hoc facies, donec summam diviseris omnem.

*Modus scribendi residuum.*

Si quid restiterit postquam divisio facta est,
Id supra scribi divisorem solet omne.
Inter et hos numeros est linea parva trahenda,
Quæ fractum numerum, non integrum notet esse.

*Rei totius brevis comprehensio.*

Divide, multiplica, subduc, transferque secantem.

*Examen.*

Per divisorem, quotientem multiplicabis.
Producto reliquum, si quod fuit, adde, priorque
Exhibit numerus, nisi te deceperit error.

---

DE EXTRACTIONE RADICIS QUADRATÆ.

Quadratæ est numerum semel in se multiplicare.
Quærere radicem, numerum est exquirere, qui in se
Ductus, propositam poterit producere summam.

Cujus radicem numeri vis quærere, scribe.
Descripti alternas punctis signato figuras,
Lineolasque duas illi supponito rectas.
Et quia præsenti similis diviso parti est,
A puncto versus lævam incipies operari,
Quærendo sub eo digitum, qui multiplicatus
In se, vel totum, vel magnam tollere partem
Signati puncto numeri possit digitusque
Sub puncto in medio spatio scribatur, et inde
In se ducatur, productum tolle supremo
Ex numero, reliquum scribens, ut quando secares

Dupletur Quotiens, producti prima figura,
Si binæ fuerint, versus dextram statuatur
Sub numero, punctum cui non supereminet ullum,
Et reliqui numeri ponantur parte sinistra.

Sic novus emergit Divisor, qui quoties sit
In supra posito numero, quæras: Quotientem
Inventum in spatio sub puncto pone sequenti.
Hunc primum in se, mox divisorem per eundem
Multiplica, producta duo summam simul unam
Efficiant, numero quæ subducenda, supremo est,
Et reliquum solito debes ascribere more.

Dupletur rursus quicquid tibi linea duplex
Suggerit, et duplum divisor erit nonus, huncque
Divide per numerum suprema parte relictum,
Cæteraque expedias quadrando, multiplicando,
Hinc subducendo, supra ponendo relictum.
Quod facies donec numeros percurreris omnes.

Si semel in reliquo duplum non possit haberi,
Pone cyphram in spatio, divisoremque novato.

*Examen.*

Quadra radicem, quadrato junge relictum,
Si modo quid fuerit, numerus si prodeat idem
Cum primo, recte est, si non opus est iterandum.

*Modos colligendi minutias ex residuo.*

Duplo radicis numerus superadditur unus,
Producto numerum mox supra scribe relictum,
Lineola adjecta numeros quæ separet ambos.

It deserves to be mentioned, that the great Napier himself did not disdain to give, in his *Rabdologia*, a short and neat set of memorial verses adapted to the use of his *Rods*.

## Note XVII.—Page 208.

The formation of circulating decimals affords a fine illustration of that secret concatenation which binds the succession of physical events, and determines the various lengthened Cycles of the returning seasons—a principle which the ancient Stoics, and some other Philosophers, have boldly extended to the moral world:

Alter erit tum Tiphys, et altera quæ vehat Argo
Delectos heroas: erunt etiam altera bella,
Atque iterum ad Trojam magnus mittetur Achilles.

THE END.